Two-scale homogenisation of partially degenerating PDEs with applications to Photonic crystals and Elasticity.

submitted by

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Contents

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Summary

In this thesis we study elliptic PDEs and PDE systems with ε-periodic coefficients, for small ε, using the theory of two-scale homogenisation. We study a class of PDEs of partially degenerating type: PDEs with coefficients that are not uniformly elliptic with respect to ε, and become degenerate in the limit $\varepsilon \to 0$. We review a recently developed theory of homogenisation for a general class of partially degenerating PDEs via the theory of two-scale convergence, and study two such problems from physics. The first problem arises from the study of a linear elastic composite with periodically dispersed inclusions that are isotropic and 'soft' in shear: the shear modulus is of order ε^2. By passing to the two-scale limit as $\varepsilon \to 0$ we find the homogenised limit equations to be a genuinely two-scale system in terms of both the macroscopic variable x and the microscopic variable y. We discover that the corresponding two-scale limit solutions must satisfy the incompressibility condition in y and therefore the composite only undergoes microscopic deformations when a 'microscopically rotational' force is applied. We analyse the corresponding limit spectral problem and find that, due to the y-incompressibility, the spectral problem is an uncoupled two-scale problem in terms of x and y. This gives a simple representation of the two-scale limit spectrum. We prove the spectral compactness result that states: the spectrum of the original operator converges to the spectrum of the limit operator in the sense of Hausdorff. The second problem we study is the propagation of electromagnetic waves down a photonic fibre with a periodic cross section. We seek solutions to Maxwell's equations, propagating down the waveguide with wave number k ε^2-close to some 'critical' value. In this setting, Maxwell's equations are reformulated as a partially degenerating PDE system with ε-periodic coefficients. Using the theory of homogenisation we pass to the limit as $\varepsilon \to 0$ to find a *non-standard* two-scale homogenised limit and prove that the spectral compactness result holds. We finally prove that there exist gaps in the limit spectrum for two particular examples: a one-dimensionally periodic 'multilayer' photonic crystal and a two-dimensionally periodic two-phase photonic crystal with the inclusion phase consisting of arbitrarily small circles. Therefore, we prove that these photonic fibres have photonic band gaps for certain k.

Chapter 1

Introduction

Partial differential equations (PDEs) with rapidly oscillating coefficients arise frequently in physics. In general, it is not possible to find an explicit solution to such problem, and numerical solutions would require a mesh grid size smaller than the oscillations of the coefficients, which can be computationally expensive. Even if it were possible to find exact solutions, in applications we do not need the solutions' every detail but some slowly varying 'local average'. One approach to such problems would be to solve an alternative related 'effective' problem whose coefficients are mildly varying, and whose solutions approximate the original solution well, in some sense. The question then becomes which alternative problem should we solve. One approach to finding the effective problem is the mathematical theory of homogenisation, which probably first originated in the work of De Giorgi and Spagnolo [16]. Homogenisation theory has been used extensively in the last several decades to find for a wide class of problems effective equations whose solutions give good approximations with a controllably small error, see for example [4, 6, 14, 33, 39].

An application of homogenisation is in the study of periodic composite materials, see Figure 1.1. Composite materials are known to display properties often not exhibited by their constitutive parts. Often we would like to predict the properties of composites without having to build them; mathematically this is a question of solving PDEs with rapidly oscillating coefficients, therefore homogenisation is a natural tool. We shall introduce the concept of Homogenisation with the following simple illustrative example: Let us consider the stationary heat field of a one-dimensional bar of length 1 composed of two distinct materials having

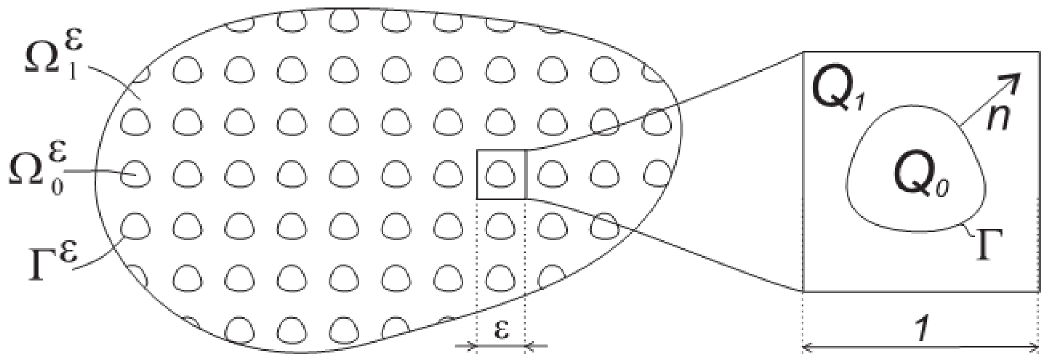

Figure 1.1: A two phase periodic composite material with periodic reference cell Q.

different specific heat capacities. The temperature at both ends of the bar is kept at zero. One material is periodically dispersed through the other on the microscopic scale (denoted by $\varepsilon \ll 1$). The temperature of the bar, u, is modelled by the following problem

$$-\frac{d}{dx}\left(a\left(\tfrac{x}{\varepsilon}\right)\frac{du}{dx}\right) = f(x), \quad u(0) = u(1) = 0, \tag{1.0.1}$$

where $f(x)$ is the macroscopic heat source and $a(y)$, in terms of the "fast variable" y, is a positive one-periodic function describing the difference in heat capacity between the constitutive parts of the composite bar. Explicitly

$$a(y) = \begin{cases} \kappa_1, & y \in \left[0, \frac{1}{2}\right), \\ \kappa_2, & y \in \left[\frac{1}{2}, 1\right). \end{cases} \tag{1.0.2}$$

The idea to find an effective, or homogenised, equation corresponding to problem (1.0.1) is to send ε to zero in (1.0.1) and ask how the behaviour of the solution depends on ε. For each ε we have a unique solution u^ε to problem (1.0.1), the question is, does this sequence u^ε converge to a function u^0 as ε tends to zero and, if so, what problem does u^0 solve? It is known, see Chapter 2, that u^ε converges

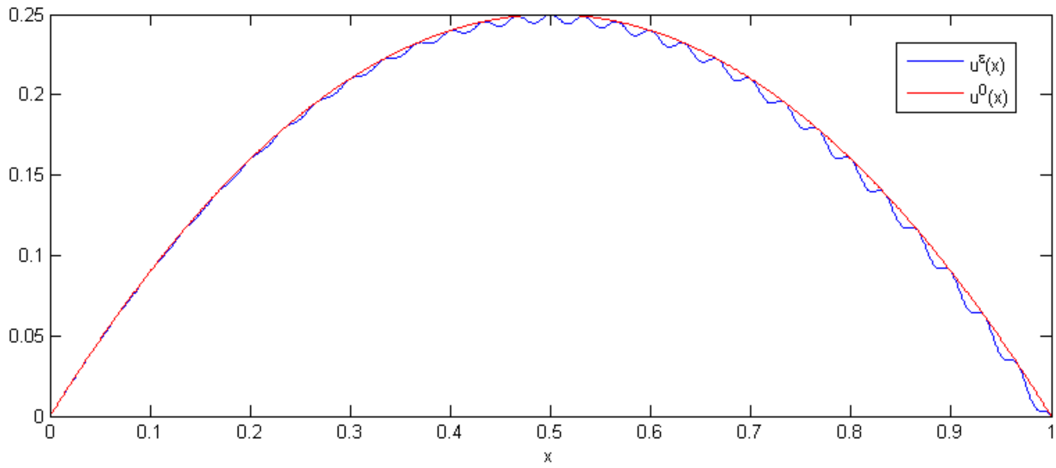

Figure 1.2: The exact and homogenised solutions for problem (1.0.1) when $a(y) = (2 + \sin(2\pi y))^{-1}$. Here $\varepsilon = 0.02$, $f(x) \equiv 1$.

to u^0, where u^0 is the solution of the following equation

$$-a^{\text{hom}} \frac{d^2 u^0}{dx^2} = f(x), \quad u(0) = u(1) = 0, \tag{1.0.3}$$

for

$$a^{\text{hom}} = \left(\int_0^1 \frac{1}{a(y)} \, \mathrm{d}y \right)^{-1}. \tag{1.0.4}$$

The homogenised equation (1.0.3) can be intuitively considered as modelling the stationary heat field in an 'equivalent' homogeneous bar, with constant heat capacitance a^{hom}. The solution u^0 approximates u^ε well, see Figure 1.2, and is known as the homogenised solution, for full details see [4, 39, 6]. Alternative proofs to the classical homogenisation problem involve passing to the limit in the corresponding weak formulation by the method of compensated compactness, see [26, 39], or two-scale convergence, see[27, 1, 37].

The above is a simplest example of classical homogenisation: the coefficients in (1.0.1) are uniformly bounded and uniformly elliptic with respect to ε, i.e.

$$0 < \nu \leq a\left(\frac{x}{\varepsilon}\right) \leq \nu^{-1}, \quad \text{for some } \nu \text{ independent of } \varepsilon. \tag{1.0.5}$$

Such problems and their homogenised limits are known to be of the same form, in particular the homogenised limit does not describe the interplay between the two, macroscopic and microscopic, scales except via (1.0.4). Composite materials that do not fall into the above class of homogenisation problem are characterised by one or more of their constitutive parts having the magnitude of their physical parameters dependent on ε, the scale of the composite's microscopic structure. Mathematically, this corresponds to loss of the validity of the uniform ellipticity condition (1.0.5). One of the first studies of problems of this nature was probably by Fenchenko and Khruslov [19]. Problems of particular interest are the so-called 'double porosity' or high contrast problems: here the physical parameters are scaled with ε^2. The technique of two-scale convergence, first developed by Nguetseng in [27], was first used by Allaire in [1] to, amongst other things, find the homogenised two-scale limit for the high contrast problems. Other classes of high contrast problems in periodic materials where stated and also considered in [29, 34].

High contrast homogenisation theory can be used to describe many non-trivial and interesting behaviours; examples include memory effects (e.g. [19, 23, 34]) and other non-standard effects (e.g. [9, 13, 12, 5, 2]), dispersion effects in wave propagation or wave localisation, and in recent years allowing a mathematical description of the properties of the so-called metamaterials. Metamaterials are man-made composite materials designed to have certain physical properties that may not be found in nature, e.g. electromagnetic materials with negative refractive index, cf. e.g. [31]. These materials are designed by constructing a, possibly periodic, small scale structure whose effective behaviour gives rise to the desired physical properties. In particular, negative refraction is related to the macroscopic effect of the so-called microresonances: microscopic inclusions with eigenfrequencies comparable with the applied 'macroscopic' frequency.

Mathematically, one may hope to capture the microresonance type behaviour of such materials by using homogenisation theory to study spectral problems of the form

$$\begin{aligned} -\mathrm{div}_x\Big(a^\varepsilon(x/\varepsilon)\nabla_x u^\varepsilon(x)\Big) &= \lambda^\varepsilon u^\varepsilon \qquad x \in \Omega, \\ u^\varepsilon(x) &= 0 \qquad x \in \partial\Omega, \end{aligned} \tag{1.0.6}$$

where $a^\varepsilon(y)$ is of the form

$$a^\varepsilon(y) = \begin{cases} a^{(1)}(y), & y \in Q_1 \\ \varepsilon^2 a^{(0)}(y), & y \in Q_0, \end{cases} \tag{1.0.7}$$

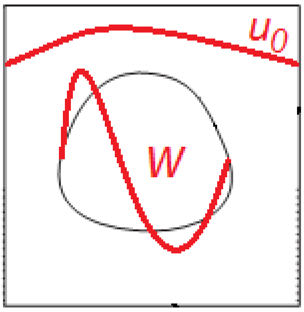

Figure 1.3: Micro-resonant effect of the inclusions described by the appearance of $w(x, y)$ due to the critical scaling $\delta = \varepsilon^2$, see Section 2.2

for uniformly elliptic $a^{(1)}(y)$, $a^{(0)}(y)$. Problems of this type where studied by Zhikov in [37] in bounded and in [38] in unbounded domains respectively. The crucial difference between (1.0.7) and the classical case (1.0.2) above, is the fact that the coefficients in (1.0.7) are high contrasting. The choice of the scaling ε^2 is important and is known as a 'critical scaling': one can see (Section 2.2) that in contrast to the classical homogenisation, this scaling results in the solution having a non-trivial dependence of the fast variable $\left(y = \frac{x}{\varepsilon}\right)$ in the inclusion phase which exactly reflects the microresonance effect of the inclusions, see Figure 1.3. That is,

$$u^\varepsilon(x) \sim \begin{cases} u_0(x) & \text{in } \Omega_1^\varepsilon \\ u_0(x) + w\left(x, \frac{x}{\varepsilon}\right) & \text{in } \Omega_0^\varepsilon \end{cases}$$

Zhikov showed in [37, 38], amongst other things, that when the inclusion phase Ω_0^ε does not intersect the boundary $\partial\Omega$, $\lambda^\varepsilon \to \lambda_0$ and the "slow" part u_0 solves

$$-\mathrm{div}_x\Big(a^{\mathrm{hom}}\nabla_x u_0(x)\Big) = \beta(\lambda_0)u_0(x), \tag{1.0.8}$$

where $\beta(\lambda)$ can be both positive and negative, see Figure 1.4, as results from the decoupling of the fast variable y and the slow variable x, see Section 2.2 for precise statements. Zhikov also showed in [37, 38] that the ranges of λ for which $\beta(\lambda)$ is negative correspond to the so-called "band gaps": physically, these correspond to the ranges of frequencies for which waves fail to propagate. In the context of time dependent problems, the properties of solutions can be revealed by knowing information about the associated spectrum, via Laplace transforms in parabolic

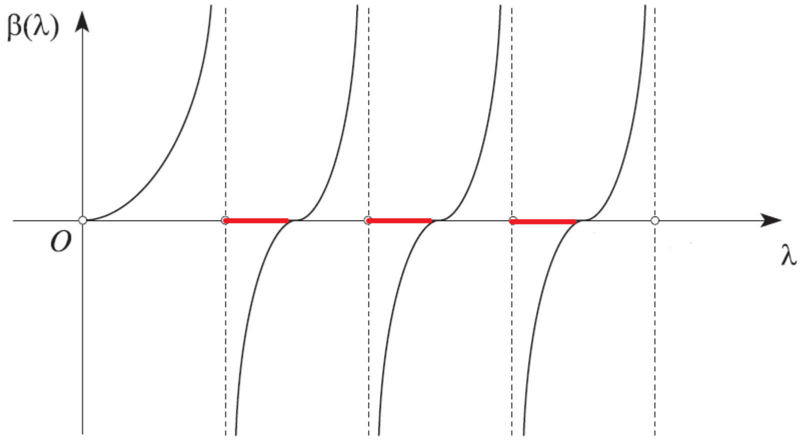

Figure 1.4: Asymptotes of $\beta(\lambda)$ are located at the eigenvalues of the Dirichlet Laplacian on Q_0 whose eigenfunctions have non-zero mean, see [37, 38].

or Fourier transforms in the hyperbolic cases. The non-linear dependence in the spectral parameter λ, described by $\beta(\lambda)$, appears to be related to time non-local effects such as memory effects, cf. for example [23, 34].

In the context of metamaterials, problem (1.0.8) can be regarded as the effective equation, for problem (1.0.6), with the effective physical parameters a^{hom} and $\beta(\lambda)$. For example, in [8], the effective behaviour of the scattering of TM polarised electromagnetic waves of a given frequency travelling through a particular dielectric medium with constant positive magnetic permeability was shown, by essentially re-discovering the results in [37, 38], to satisfy a problem of the form (1.0.8) with the sign changing $\beta(\lambda)$. Furthermore, in [8], $\beta(\lambda)$ was for the first time interpreted as an effective magnetic permeability, which is allowed to be negative for certain wave frequencies, see Figure 1.4.

We see that the desired property of metamaterials, i.e. the material's physical properties being strikingly different for waves of certain frequencies, can be achieved by constructing a periodic material with highly contrasting coefficients. A natural question that follows, and was pursued in [35] via formal asymptotics in the context of elasticity, is what if the coefficients of $a(y)$ are not fully high contrasting between the material phases, but have partial high contrasts. For example, if we consider

$$a^{\varepsilon}(y) = \begin{cases} a^{(1)}(y), & y \in Q_1 \\ a^{(2)}(y) + \varepsilon^2 a^{(0)}(y), & y \in Q_0, \end{cases}$$

for uniformly elliptic $a^{(1)}(y)$, $a^{(0)}(y)$ and non-negative $a^{(2)}(y)$. Such an $a^{\varepsilon}(y)$ can be called partially highly contrasting due to the presence of the $a^{(2)}$ term: if one formally sets $\varepsilon = 0$, $a(y)$ is no longer identically zero in Q_0 but a degenerate matrix $a^{(2)}(y)$. In [35], theoretical elastic materials of this type were constructed that not only exhibit the (frequency) band gaps described above but the materials where shown to display the phenomenon of 'directional gaps': waves of certain frequencies are forbidden to propagate in certain directions through the material.

The main focus and original contribution of this thesis is to consider two physical problems that lead to PDEs with partially high contrasting coefficients, and to review but also appropriately develop further the general theory of homogenisation for partially degenerating PDEs. This thesis has the following structure: Chapter 2 is dedicated to a brief review of the relevant classical and high contrast homogenisation theory. Section 2.1 will review the standard classical periodic homogenisation results via the method of multiscale asymptotics. Section 2.2 reviews the periodic high-contrast homogenisation results in the context of two-scale convergence. Also we discuss the fact that the choice of scaling $\delta(\varepsilon)$ between the coefficients of the inclusion and matrix phases results in a non-trivial genuinely two-scale homogenised limit only when $\delta \sim \varepsilon^2$. Chapter 3 reviews the recent development of two-scale homogenisation of partially degenerating PDEs, [21]. In this chapter we shall present the recent modifications to two-scale homogenisation which allows one to rigorously pass to the two-scale limit of 'resolvent' PDEs with a general class of partial degeneracies. In Chapter 4 we consider an elastic composite whose inclusion phase is an isotropic elastic material with a critically scaled shear modulus. The two-scale homogenised limit of this problem is rigorously justified, we perform a detailed analysis of the homogenised limit, study its spectrum and prove spectral compactness results. We find that the homogenised limit elastostatic equations have the novel feature of their structure being of a classical or two-scale limit form depending upon the microscopic nature of the external body force. Such properties have interesting consequences, especially when looking at the limit spectrum. For this particular partially degenerate problem, the homogenised limit is of a genuine two-scale nature but the associated spectrum has no gaps. In Chapter 5 by considering propagating electromagnetic waves, with a wave number close to a certain critical value, down a *mildly* contrasting dielectric photonic crystal we naturally arrive at a *partially high contrasting* PDE. The coefficients of this PDE have the novel feature of being asymptotically partially degenerate in the whole of Q. As a consequence the two-scale homogenised limit function is shown to have a non-trivial Bloch decomposition. We rigorously justify the two-scale homogenised limit, show the homogenised limit spectrum to have a band structure as well as prove the spectral compactness result. Furthermore, for the one dimensional multilayer photonic crystal, and a particular two dimensional crystal with arbitrarily small inclusion phase, we show that there exist gaps in the spectrum of the related limit operator and therefore, due to the spectral compactness, there exist gaps in the spectrum of the original problem for small enough values of a parameter ε. Chapter 6 contains conclusions and highlights some of the possible further developments to the homogenisation of partially degenerating PDEs which have been observed during the original study presented in Chapters 4 and 5. Appendix A lists the mathematical nomenclature used in this thesis. Appendix B reviews the two-scale convergence and its main properties. Appendix C contains known results from Functional Analysis that are used in the thesis but not solely relevant to homogenisation theory.

Chapter 2

Homogenisation of second order elliptic PDEs

This chapter is dedicated to a brief review of the standard periodic homogenisation theory of second order elliptic partial differential equations as well as of the 'fully' high contrast homogenisation theory. We do not attempt to give here a comprehensive literature review of homogenisation theory. For classical homogenisation, detailed reviews and extensive reference lists can be found in e.g. [6, 4, 39, 14].

Throughout the chapter we shall consider an open bounded domain $\Omega \subset \mathbb{R}^d$ to have the following geometric structure, see Figure 1.1. Let the unit cube $Q = [0,1)^d$ be separated into two regions: $Q_0 \subset\subset Q$ a bounded set strictly included in Q with smooth boundary Γ and $Q_1 = Q\backslash\overline{Q_0}$ the complement of $\overline{Q_0}$. We assume Q_1 to be connected. We denote by F_0 the 1-periodic extension of Q_0 throughout $\mathbb{R}^d$:

$$F_0 := \left\{x : \ x = y + k \text{ for some } y \in Q_0 \text{ and some } k \in \mathbb{Z}^d\right\},$$

and assume $F_1 := \mathbb{R}^d\backslash F_0$ to be connected. Denote by εF_0 the ε contraction of F_0, i.e. $\varepsilon F_0 = \{x : x/\varepsilon \in F_0\}$. Then $\Omega_0^\varepsilon = \Omega \cap \varepsilon F_0$ is the 'inclusion' phase and $\Omega_1^\varepsilon = \Omega\backslash\overline{\Omega_0^\varepsilon}$ is the 'matrix' phase; see Figure 1.1.

In this thesis we shall consider further generalisations of problems of the following form: For fixed $0 < \varepsilon \ll 1$, find $u^\varepsilon \in H_0^1(\Omega)$ the solution to

$$-\mathrm{div}\Big(a^\varepsilon(x/\varepsilon)\nabla u^\varepsilon(x)\Big) + \alpha u^\varepsilon(x) = f^\varepsilon(x) \qquad x \in \Omega. \tag{2.0.1}$$

Here $\alpha \geq 0$, $f^\varepsilon \in L^2(\Omega)$ and bounded positive $a^\varepsilon(y) \in [L^\infty_\#(Q)]^{d\times d}$ are known. The case $\alpha > 0$ corresponds to a resolvent problem for the elliptic operator $A^\varepsilon u = -\mathrm{div}\,(a\nabla u)$. The case $\alpha = 0$ is a classical "static" problem. Understanding the resolvent problem allows one to understand the spectral properties of A^ε as $\varepsilon \to 0$ and hence the properties of time-dependent problems. We are interested in the behaviour of the solution u^ε as $\varepsilon \to 0$. Clearly this behaviour depends on the asymptotic behaviour of $a^\varepsilon(x/\varepsilon)$ as $\varepsilon \to 0$. We shall review two general classes of $a^\varepsilon(y)$ in this chapter. In Section 2.1, we review the so called classical homogenisation theory, where $a^\varepsilon(y)$ is considered to be uniformly elliptic with respect to ε, via the method of multiscale asymptotics. In Section 2.2 the high contrast case is considered, where $a^\varepsilon(y)$ asymptotically 'degenerates' in the inclusion phase: $a \sim O(\delta(\varepsilon))$ for some $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Here we shall review the standard high contrast homogenisation theory via the method of two-scale convergence, see Appendix B , and its applications to wave propagation and localisation.

2.1 Classical Homogenisation

Let us consider the following 'resolvent' problem:

$$-\mathrm{div}_x\Big(a(x/\varepsilon)\nabla u^\varepsilon(x)\Big) + \alpha u^\varepsilon(x) = f^\varepsilon(x) \qquad x \in \Omega. \tag{2.1.1}$$

Here, $\alpha \geq 0$, $f^\varepsilon(x) = f(x, \frac{x}{\varepsilon})$ for a given sufficiently smooth $f(x,y)$ that is Q-periodic with respect to y. The symmetric matrix $a(y) \in [L^\infty(Q)]^{d\times d}$ is uniformly bounded and uniformly elliptic with respect to ε, that is $\exists\ \nu > 0$ independent of ε such that

$$|\eta|^2\nu \leq a(y)\eta\cdot\eta \leq \nu^{-1}|\eta|^2, \qquad \forall \eta \in \mathbb{R}^d,\ \forall y \in Q. \tag{2.1.2}$$

There are several ways to arrive at the homogenised limit for problem (2.1.1).

In this chapter we shall formally arrive at the homogenised limit via the method of formal multiscale asymptotic expansions. In Section 3.2 we shall present a new proof to the classical homogenisation result using the newly developed tools presented therein. The method of multi-scale asymptotic expansions seeks a formal asymptotic expansion to the solution of (2.1.1) of the form

$$u^\varepsilon(x) \sim u^0(x, \tfrac{x}{\varepsilon}) + \varepsilon u^1(x, \tfrac{x}{\varepsilon}) + \varepsilon^2 u^2(x, \tfrac{x}{\varepsilon}) + \dots, \tag{2.1.3}$$

where $u^i(x, y)$ are Q-periodic with respect to the y variable.
Substituting (2.1.3) into (2.1.1) and equating the coefficients for the powers of ε gives the following system of equations:

$$-\mathrm{div}_y\Big(a(y)\nabla_y u^0(x,y)\Big) = 0, \tag{2.1.4}$$

$$-\mathrm{div}_y\Big(a(y)\nabla_y u^1(x,y)\Big) = \mathrm{div}_y\Big(a(y)\nabla_x u^0(x,y)\Big) + \mathrm{div}_x\Big(a(y)\nabla_y u^0(x,y)\Big), \tag{2.1.5}$$

$$-\mathrm{div}_y\Big(a(y)\nabla_y u^2(x,y)\Big) = F(x,y), \tag{2.1.6}$$

where

$$F(x,y) := \mathrm{div}_x\Big(a(y)\left(\nabla_x u^0(x,y) + \nabla_y u^1(x,y)\right)\Big) + \\ + \mathrm{div}_y\Big(a(y)\nabla_x u^1(x,y)\Big) - \alpha u^0(x,y) + f(x,y). \tag{2.1.7}$$

Since $a(y)$ satisfies (2.1.2), (2.1.4) implies that u^0 is independent of y, i.e. $u^0(x,y) = u(x)$. We can then see from (2.1.5) that $u^1(x,y) = N_r(y)u_{,r}(x)$ where, for $r = 1, \dots, n$, $N_r(y)$ solve the so-called cell problem:

$$-\mathrm{div}_y\Big(a(y)\nabla_y N_r(y)\Big) = \mathrm{div}_y\Big(a(y)e_r\Big),$$

here e_r is the r$^{\text{th}}$ Euclidean basis vector. We finally note that problem (2.1.6) has a (weak) solution in $H_0^1(\Omega)$ if, and only if, the mean value of $F(x,y)$ with respect to y is zero, i.e. $\langle F\rangle(x) = 0$; this solvability condition and (2.1.7) give rise to the following problem for $u(x)$:

$$-\mathrm{div}_x\Big(a^{\mathrm{hom}}\nabla_x u(x)\Big) + \alpha u(x) = \langle f\rangle(x), \tag{2.1.8}$$

where a^{hom} is the symmetric constant coefficient matrix given by

$$a_{ij}^{\text{hom}} = \int_Q a_{ik}(y) \left(N_{j,k}(y) + \delta_{jk} \right) \, \mathrm{d}y \quad . \tag{2.1.9}$$

with δ_{jk} being the Kronecker delta. Furthermore it can be shown, see e.g. [10], that for $\eta \in \mathbb{R}^d$:

$$a^{\text{hom}} \eta \cdot \eta = \inf_{v \in H_\#^1(Q)} \int_Q a(y)(\nabla_y v(y) + \eta) \cdot (\nabla_y v(y) + \eta) \, \mathrm{d}y. \tag{2.1.10}$$

The variational characterisation (2.1.10) allows us to see that a^{hom} is uniformly bounded and uniformly elliptic, satisfying (2.1.2). Equation (2.1.8) is the homogenised equation for problem (2.1.1) and $u \in H_0^1(\Omega)$ is the homogenised solution. It is a good place to remark here that the main observation is that problem (2.1.8) depends only on the x variable and this is due to the uniform ellipticity of $a(y)$.

2.2 High contrast Homogenisation

In this section the matrix $a^\varepsilon(y)$ shall asymptotically behave, in the inclusion phase Q_0, like δ, where the second small parameter $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ is to be determined. It turns out that there is a critical scaling $\delta = \varepsilon^2$ which, as we will see, is the only scaling that results in the homogenised equation being a non-trivial two-scale problem in terms of the macroscopic and microscopic variables. An insightful way to see why the scaling $\delta = \varepsilon^2$ is special is to consider the formal multiscale expansion, to problem (2.1.1), for

$$a^\varepsilon(y) = \begin{cases} I, & y \in Q_1 \\ \delta(\varepsilon) I, & y \in Q_0, \end{cases}$$

of the form

$$u^\varepsilon(x) \sim u^0(x, \tfrac{x}{\varepsilon}) + \varepsilon u^1(x, \tfrac{x}{\varepsilon}) + \varepsilon^2 u^2(x, \tfrac{x}{\varepsilon}) + \dots .$$

As in the previous section, equating coefficients of ε^{-2} gives rise to the following characterisation of the limit function u^0:

$$-\Delta_y u^0(x,y) = 0 \qquad\qquad y \in Q_1,$$

Unlike in the classical case, we find at this stage that the homogenised limit function $u^0(x,y)$ is independent of y in Q_1 but so far arbitrary in Q_0, i.e. $u^0(x,y) = u(x) + \chi_0(y)v(x,y)$. Proceeding as in Section 2.1, equating coefficients of ε^0 gives rise to the homogenised equation for u^0; now it is clear that unless $\delta = \varepsilon^2$ the function v will not appear in the homogenised limit equation and therefore have no bearing on the behaviour of $u(x)$, resulting in a classical homogenisation theorem, i.e. a homogenised limit equation depending only on the macroscopic variable x. We shall now present these arguments rigorously via the method of two-scale convergence, see Appendix B, in the case $\Omega \subset \mathbb{R}^d$, $d \geq 2$. The case $d = 1$ is different and will not be considered here.

For fixed $\varepsilon \ll 1$, fixed $\delta(\varepsilon) > 0$, let u^ε be the solution to

$$\begin{aligned} -\text{div}_x\Big(a^\varepsilon(x/\varepsilon)\nabla_x u^\varepsilon(x)\Big) + u^\varepsilon(x) = f^\varepsilon(x) \qquad & x \in \Omega, \\ u^\varepsilon(x) = 0 \qquad & x \in \partial\Omega, \end{aligned} \tag{2.2.1}$$

where $a^\varepsilon(y)$ is of the form

$$a^\varepsilon(y) = \begin{cases} a^{(1)}(y), & y \in Q_1 \\ \delta(\varepsilon)a^{(0)}(y), & y \in Q_0. \end{cases} \tag{2.2.2}$$

Here the symmetric $a^i(y)$ are uniformly elliptic, uniformly bounded matrices, i.e. there exists $\nu > 0$ independent of ε such that, for $i = 0, 1$

$$a^i(y)\eta \cdot \eta \geq \nu|\eta|^2, \qquad\qquad \forall \eta \in \mathbb{R}^d,\ \forall y \in Q. \tag{2.2.3}$$

$a^\varepsilon(y)$ is an example of a matrix that is elliptic but becomes degenerate in Q_0 in the limit $\varepsilon \to 0$. We say such a matrix $a^\varepsilon(y)$ fully degenerates in Q_0 as $\varepsilon \to 0$. $\delta(\varepsilon)$ has to, up to a subsequence, be considered to asymptotically behave in one of the following three essentially exhaustive cases

$$\text{case (i):} \qquad \delta(\varepsilon) = \varepsilon^2, \tag{2.2.4}$$

$$\text{case (ii):} \qquad \varepsilon^{-2}\delta(\varepsilon) \to 0 \text{ as } \varepsilon \to 0, \tag{2.2.5}$$

$$\text{case (iii):} \qquad \varepsilon^{-2}\delta(\varepsilon) \to \infty \text{ as } \varepsilon \to 0. \tag{2.2.6}$$

Problem (2.2.1) has the weak formulation: find $u^\varepsilon \in H_0^1(\Omega)$ such that

$$\int_{\Omega_1^\varepsilon} a^{(1)}(x/\varepsilon)\nabla u^\varepsilon(x) \cdot \nabla\phi(x)\ \mathrm{d}x \ + \ \delta(\varepsilon)\int_{\Omega_0^\varepsilon} a^{(0)}(x/\varepsilon)\nabla u^\varepsilon(x) \cdot \nabla\phi(x)\ \mathrm{d}x \ + $$
$$+ \int_\Omega u^\varepsilon(x)\phi(x)\ \mathrm{d}x = \int_\Omega f^\varepsilon(x)\phi(x)\ \mathrm{d}x, \quad \forall\phi \in H_0^1(\Omega). \tag{2.2.7}$$

We find, by the uniform ellipticity assumption (2.2.3), setting $\phi = u^\varepsilon$ in (2.2.7) gives rise to the following a priori estimates: there exists a constant C independent of ε such that

$$\|u^\varepsilon\|_{L^2(\Omega)} \le C\|f^\varepsilon\|_{L^2(\Omega)}, \tag{2.2.8}$$

$$\|\nabla u^\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \le C\|f^\varepsilon\|_{L^2(\Omega)}, \tag{2.2.9}$$

$$\delta^{1/2}(\varepsilon)\|\nabla u^\varepsilon\|_{L^2(\Omega_0^\varepsilon)} \le C\|f^\varepsilon\|_{L^2(\Omega)}. \tag{2.2.10}$$

The inequalities (2.2.8)-(2.2.9) and the two-scale compactness theorem imply the following, see [1, 37],

Lemma 2.2.1. *There exist $u \in H_0^1(\Omega)$, $v \in L^2(\Omega \times Q)$ such that*

$$u^\varepsilon \stackrel{2}{\rightharpoonup} u(x) + \chi_0(y)v(x,y),$$
$$\int_{\Omega_1^\varepsilon} a^{(1)}\nabla u^\varepsilon \cdot \nabla\phi(x)\ \mathrm{d}x \longrightarrow \int_\Omega a^{hom}\nabla_x u(x) \cdot \nabla_x\phi(x)\ \mathrm{d}x, \quad \forall\phi \in C_0^\infty(\Omega)$$

where a^{hom} is the constant positive symmetric homogenised matrix for a perforated domain (with 'pores' at Q_0). That is,

$$a_{ij}^{hom} = \int_{Q_1} a_{ip}^{(1)}(y)(N_{,p}^j(y) + \delta_{jp})\ \mathrm{d}y.$$

$N^j(y) \in H^1_\#(Q_1)$ are the weak solutions to

$$\int_{Q_1} a^{(1)}(y)\left(\nabla_y N^j(y) + e^j\right) \cdot \nabla_y \phi = 0, \quad \forall \phi \in H^1_\#(Q).$$

The asymptotic behaviour of $\delta(\varepsilon)$ determines the behaviour of the homogenised limit equation. We shall see for the cases (ii) and (iii) the limit equations dependence on the microscopic variable y is trivial, see [37].

Theorem 2.2.2. *Let the behaviour of $\delta(\varepsilon)$ be given by (2.2.5). Then $u^\varepsilon \xrightarrow{2} u(x) + \chi_0(y)v(x,y)$, where (u,v) is the solution to*

$$\begin{aligned} v(x,y)|_{Q_0} &= f(x,y)|_{Q_0} \\ -\mathrm{div}_x\Big(a^{hom}\nabla_x u(x)\Big) + u(x) + \langle v\rangle_{Q_0}(x) &= \langle f\rangle(x) \end{aligned}$$

Theorem 2.2.3. *Let the behaviour of $\delta(\varepsilon)$ be given by (2.2.6). Then $u^\varepsilon \to u$ strongly in L^2 where u is the solution of*

$$-\mathrm{div}_x\Big(a^{hom}\nabla_x u(x)\Big) + u(x) = \langle f\rangle(x).$$

For the case (2.2.5) the homogenisation theorem, see [1, 37], states

Theorem 2.2.4. *Let $\delta(\varepsilon) = \varepsilon^2$. Let f^ε strongly (weakly) two-scale converge to $f(x,y)$. Then, up to a subsequence, u^ε strongly (weakly) two-scale converges to $u(x) + \chi_0(y)v(x,y)$ where $(u,v) \in H^1_0(\Omega) \times L^2(\Omega; H^1_0(Q_0))$ is the unique solution to*

$$\begin{aligned} -\mathrm{div}_x\Big(a^{hom}\nabla_x u(x)\Big) + u(x) + \langle v\rangle &= \langle f\rangle(x) && x \in \Omega, \\ -\mathrm{div}_y\Big(a^{(0)}(y)\nabla_y v(x,y)\Big) + v(x,y) + u(x) &= f(x,y) && y \in Q_0, \\ v(x,y) &= 0 && y \in \Gamma. \end{aligned}$$

Theorem 2.2.2 and Theorem 2.2.3 tell us that for the cases (2.2.5), (2.2.6) the homogenised limit is either a one-scale problem or trivially depends on the second scale. To the contrary, we see by Theorem 2.2.4, that the critical scaling $\delta = \varepsilon^2$ results in the homogenised limit equation being of a genuinely nontrivial two-scale nature. We will now show how the presence of the microscopic oscillations

$v(x, y)$ can be used to describe phenomena such as wave propagation and wave localisation.

Assume we are looking for propagating waves with frequency ω through the composite material occupying the whole space $\Omega = \mathbb{R}^d$, i.e. solutions $w^\varepsilon(x, t) = e^{-i\omega t} u^\varepsilon(x)$ to

$$\frac{\partial^2 w^\varepsilon}{\partial t^2}(x, t) = \operatorname{div}_x\Big(a(x/\varepsilon)\nabla_x w^\varepsilon(x, t)\Big).$$

Then we say that the wave, with frequency ω, propagates if we find a non-trivial bounded solution to

$$-\operatorname{div}_x\Big(a(x/\varepsilon)\nabla_x u^\varepsilon(x)\Big) = \omega^2 u^\varepsilon(x),$$

i.e. $\lambda = \omega^2$ is in the spectrum $\sigma(A^\varepsilon)$ of the operator $A^\varepsilon := -\operatorname{div}_x\left(a\left(\frac{x}{\varepsilon}\right)\nabla_x\right)$. Waves of frequency ω are forbidden to propagate if there are gaps in the spectrum $\sigma(A^\varepsilon)$, that is if $\lambda = \omega^2 \in (0, \infty)\backslash\sigma(A^\varepsilon)$. Such examples motivate the question: what is the spectrum of A^ε? It is known, see [37, 38], that the spectrum $\sigma(A^\varepsilon)$ converges in the sense of Hausdorff to the spectrum $\sigma(A)$ of the two-scale homogenised limit operator A given by Theorem 2.2.4. That is

1. For every $\lambda \in \sigma(A)$ there exists $\lambda_\varepsilon \in \sigma(A_\varepsilon)$ such that $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to 0$.

2. If there exists $\lambda_\varepsilon \in \sigma(A_\varepsilon)$ such that $\lambda_\varepsilon \to \lambda$, as $\varepsilon \to 0$, then $\lambda \in \sigma(A)$.

This important property about the spectra tells us that for small enough ε to find the spectrum of A^ε it is sufficient to study the spectrum of the homogenised limit operator A. In particular, we know that $\sigma(A^\varepsilon)$ will have gaps for small enough ε, if $\sigma(A)$ is shown to have gaps.

To study the spectrum $\sigma(A)$ we study, formally, the spectral problem $Aw = \lambda w$ which, by Theorem 2.2.4 corresponds to: find $w(x, y) = u(x) + v(x, y)$ such that

$$-\operatorname{div}_x\Big(a^{\text{hom}}\nabla_x u(x)\Big) = \lambda u(x) + \lambda\langle v\rangle(x) \qquad x \in \mathbb{R}^d, \tag{2.2.11}$$

$$-\operatorname{div}_y\Big(a^{(0)}(y)\nabla_y v(x, y)\Big) = \lambda u(x) + \lambda v(x, y) \qquad y \in Q_0, \tag{2.2.12}$$

$$v(x, y) = 0 \qquad y \in \Gamma. \tag{2.2.13}$$

Seek a solution to (2.2.12)-(2.2.13) in the form $v(x, y) = \lambda u(x)b(y)$, for given

$u(x)$. Then $b(y)$ is the solution to

$$\begin{aligned} -\mathrm{div}_y\Big(a^{(0)}(y)\nabla_y b(y)\Big) &= \lambda b(y) + 1 \quad && y \in Q_0, \\ b(y) &= 0 && y \in \Gamma. \end{aligned} \tag{2.2.14}$$

We find, see (2.2.11), that $u(x)$ must necessarily solve

$$-\mathrm{div}_x\Big(a^{\mathrm{hom}}\nabla_x u(x)\Big) = \beta(\lambda)u(x) \qquad x \in \mathbb{R}^d \tag{2.2.15}$$

for $\beta(\lambda) := \lambda + \lambda^2\langle b\rangle$. Applying the spectral decomposition for $b(y)$ solving (2.2.14), we conclude that

$$b(y) = \sum_{n=1}^{\infty} \frac{\langle\varphi_n\rangle}{\lambda_n - \lambda}\varphi_n(y), \quad \text{in } Q_0, \quad \lambda \neq \lambda_n,$$

where $(\lambda_n, \varphi_n(y))$ is the eigenvalue-eigenfunction pair of the Dirichlet Laplacian in Q_0. Hence

$$\beta(\lambda) = \lambda + \lambda^2 \sum_{n=1}^{\infty} \frac{\langle\varphi_n\rangle^2}{\lambda_n - \lambda},$$

We notice, see Figure 1.4, that $\beta < 0$ for certain values of λ. It is well known that for positive a^{hom} the spectrum of $A^0 := -\mathrm{div}_x(a^{\mathrm{hom}}\nabla_x)$ is essential (in fact continuous) and occupies $[0, \infty)$ and therefore $\lambda \notin \sigma(A)$ when $\beta(\lambda) < 0$. In fact $\beta(\lambda)$ allows us to fully characterise the spectrum of A, see [37, 38] for details. In particular the bands where β is positive are the essential part of the spectrum $\sigma(A)$ and the eigenvalues, if they exist, are the eigenvalues of the Dirichlet Laplacian on Q_0 whose corresponding eigenfunctions have zero mean.

Chapter 3

Homogenisation of partially degenerating PDE and PDE systems

In Section 2.2 we saw that the two-scale homogenised limit for a high contrast problem, with the critical scaling $\delta = \varepsilon^2$, is of a genuine two-scale nature. Furthermore, it was shown that the presence of microscopic resonances can lead to physical phenomena such as propagation and localisation of waves. Building on this idea, in [35], Smyshlyaev showed, via formal asymptotic solutions, that for an elastic material with a partial high contrasting elasticity tensor C of the form

$$C(y) = \begin{cases} C^{(1)}(y), & y \in Q_1, \\ C^{(2)}(y) + \varepsilon^2 C^{(0)}(y), & y \in Q_0, \end{cases}$$

it was possible for such materials, with 'interconnected' Q_0, to exhibit the phenomena of directional localisation, i.e. for waves of certain frequencies being only allowed to propagate in certain directions. This inspired the rigorous study of elliptic operators with partially degenerating tensors. This chapter is dedicated to the recent developments of the homogenisation of partially degenerating PDEs. In Section 3.1 we shall present the newly developed tools and use them to prove the homogenisation theorem, all results in the section were first proved by Kamotski and Smyshlyaev in [21]. In Section 3.2 we will present new proofs to the classical and high contrast homogenisation theorems from the perspective of partially degenerating PDEs.

3.1 Resolvent problem formulation and the homogenisation theorem

In this section we shall consider general elliptic systems in the following 'resolvent' form

$$-\operatorname{div}\Big(a^\varepsilon(x)\nabla u\Big) \,+\, \alpha\,\rho^\varepsilon(x)\,u \,=\, f^\varepsilon(x). \tag{3.1.1}$$

The domain Ω can be both bounded and unbounded (in particular $\Omega = \mathbb{R}^d$). Here $u \in \left(H_0^1(\Omega)\right)^n$, $n \geq 1$ is the sought (possibly vector-valued) function, $\alpha > 0$ is a real positive parameter, $\varepsilon > 0$ is a small parameter. The "forcing" $f^\varepsilon : \Omega \to \mathbb{R}^n$ is generally bounded in $\left(L^2(\Omega)\right)^n$, uniformly in ε. The "density" $\rho^\varepsilon(x)$ is generally assumed an ε-periodic bounded and uniformly positive matrix-valued function: there exists constant $\nu > 0$ such that

$$\rho^\varepsilon(x) = \rho\left(\tfrac{x}{\varepsilon}\right), \quad \rho \in [L^\infty_\#(Q)]^{n\times n}, \quad \rho_{ij}(y)\xi_i\xi_j \geq \nu|\xi|^2 \geq 0, \quad \forall y \in Q, \forall \xi \in \mathbb{R}^n.$$

The elliptic tensor is of the form

$$a^\varepsilon(x) \,=\, a^{(1)}\left(\frac{x}{\varepsilon}\right) \,+\, \varepsilon^2\, a^{(0)}\left(\frac{x}{\varepsilon}\right), \tag{3.1.2}$$

where, for $l = 1, 0$,

$$a^{(l)} \in \left(L^\infty_\#(Q)\right)^{n\times d\times n\times d}, \tag{3.1.3}$$

are symmetric: $a^{(l)}_{ijpq}(y) \,=\, a^{(l)}_{pqij}(y)$, $\forall 1 \leq i, p \leq n$, $1 \leq j, q \leq d$, $\forall y \in Q$. The tensor $a^{(1)}$ is further assumed to be non-negative, i.e.

$$a^{(1)}_{ijpq}(y)\zeta_{ij}\zeta_{pq} \,\geq\, 0, \quad \forall \zeta \in \mathbb{R}^{n\times d}. \tag{3.1.4}$$

The tensor $a^{(0)}$ is in turn assumed to be such that $a^{(0)} + a^{(1)}$ is uniformly strongly elliptic, in the sense that, as a quadratic form, it is bounded from below by a constant uniformly coercive tensor $A^{(0)}$:

$$
\begin{aligned}
\left(a^{(0)}_{ijpq}(y) + a^{(1)}_{ijpq}(y)\right)\zeta_{ij}\zeta_{pq} \geq A^{(0)}_{ijpq}\zeta_{ij}\zeta_{pq}, \qquad & \forall \zeta \in \mathbb{R}^{n\times d}, \\
A^{(0)}_{ijpq}\xi_i\eta_j\xi_p\eta_q \geq \nu|\xi|^2|\eta|^2, \qquad & \forall \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^d,
\end{aligned} \tag{3.1.5}
$$

with some coercivity constant $\nu > 0$.
Note that the 'fully' degenerating and scalar elliptic homogenisation problems considered in the earlier sections are included in this more general setting. Namely, for $n = 1$, for $a^{(0)}(y) \equiv 0$ and $a^{(1)}(y)$ uniformly positive, i.e. $a^{(1)} > \nu I$ we are in the classical regime, while if

$$
a^{(1)}(y) = \begin{cases} a^{(1)}(y) > \nu I, & y \in Q_1, \\ 0, & y \in Q_0, \end{cases}
$$

then we are in the high contrast regime. For the more general tensor (3.1.2) and for $a^{(1)}(y)$ not uniformly positive we say the tensor is partially degenerating. The boundary value problem has following weak formulation: Find $u^\varepsilon \in [H^1_0(\Omega)]^n$ such that

$$
\int_\Omega \left[a^\varepsilon(x)\nabla u^\varepsilon \cdot \nabla\phi(x) + \alpha\,\rho^\varepsilon(x)\,u^\varepsilon \cdot \phi(x)\right]\,\mathrm{d}x = \int_\Omega f^\varepsilon(x)\cdot\phi(x)\,\mathrm{d}x, \qquad \forall \phi \in [H^1_0(\Omega)]^n. \tag{3.1.6}
$$

From standard theory, Lax-Milgram Lemma guarantees the existence and uniqueness of a solution to problem (3.1.6). This can be seen by trivially extending functions from $H^1_0(\Omega)$ to $\mathbb{R}^d$, then applying Fourier transforms and (3.1.5) implies coercivity and boundedness of the bilinear form defined by the left hand side of (3.1.6).

First we wish to study the behaviour of u^ε as ε tends to zero. For fixed ε, choosing a test function $\phi = u^\varepsilon$ in (3.1.6) we arrive at the following a priori bounds

$$
\begin{aligned}
\|u^\varepsilon\|_{L^2(\Omega)} &\leq C\|f^\varepsilon\|_{L^2(\Omega)} \\
\|\varepsilon\nabla u^\varepsilon\|_{L^2(\Omega)} &\leq C\|f^\varepsilon\|_{L^2(\Omega)} \\
\|(a^{(1)})^{1/2}\nabla u^\varepsilon\|_{L^2(\Omega)} &\leq C\|f^\varepsilon\|_{L^2(\Omega)},
\end{aligned} \tag{3.1.7}
$$

for some constant C independent of ε. Here $\left(a^{(1)}\right)^{1/2}$ is well defined since $a^{(1)}$

is symmetric and non-negative, (3.1.4). By considering a two-scale convergent sequence f^ε we see that sequences in (3.1.7) are bounded and have two-scale convergent subsequences. In particular, using the tools of the theory of two-scale convergence, see Appendix B,

Lemma 3.1.1. *There exist $u_0(x,y) \in L^2(\Omega; V)$ and $\xi_0(x,y) \in L^2(\Omega; W)$ such that, up to extracting a subsequence in ε which we do not relabel,*

$$u^\varepsilon \overset{2}{\rightharpoonup} u_0(x,y) \tag{3.1.8}$$

$$\varepsilon\nabla u^\varepsilon \overset{2}{\rightharpoonup} \nabla_y u_0(x,y) \tag{3.1.9}$$

$$\big(a^{(1)}(x/\varepsilon)\big)^{1/2}\nabla u^\varepsilon \overset{2}{\rightharpoonup} \xi_0(x,y). \tag{3.1.10}$$

Here

$$V := \left\{v \in [H^1_\#(Q)]^n : a^{(1)}(y)\nabla_y v(y) = 0\right\}, \tag{3.1.11}$$

and

$$W := \left\{ \psi \in \left(L^2_\#(Q)\right)^{n\times d} \,\middle|\, div_y\left(\left(a^{(1)}(y)\right)^{1/2}\psi(y)\right) = 0 \;\; in \;\; \left(H^{-1}_\#(Q)\right)^n \right\}. \tag{3.1.12}$$

V is called the space of microscopic oscillations as it is the collection of admissible functions describing the behaviour of the limit function u_0 with respect the microscopic variable y; it therefore describes, in some sense, the possible microscopic behaviour of u^ε. Indeed, in classical homogenisation, since $a^{(1)} > 0$, $a^{(1)}\nabla_y v = 0$ implies $\nabla_y v = 0$ and therefore V is the space of constant functions. Likewise, in the high contrast regime, v is constant in Q_1 as $a^{(1)} > 0$ in Q_1 but v is arbitrary in Q_0 due to the full degeneracy of $a^{(1)}$ in Q_0; i.e.

$$V := \left\{v \in H^1_\#(Q) : v = c + w, \text{for some constant } c \text{ and } w \in H^1_0(Q_0)\right\}.$$

Similarly, the 'dual' space W, defined by (3.1.12), can be viewed as a space of admissible 'microscopic fluxes'.

Proof of Lemma 3.1.1. According to the theorem on relative (weak) two-scale compactness of a bounded sequence in $L^2(\Omega)$, see e.g. [27, 1] and Lemma B.0.2 (i) of Appendix B, the a priori estimates (3.1.7) imply, up to extracting a subsequence in ε (not relabelled), the existence of weak two-scale limits $\xi_0 \in [L^2(\Omega \times Q)]^{n\times d} = \left[L^2\left(\Omega;\, L^2_{\#}(Q)\right)\right]^{n\times d}$, which yields (3.1.10).

We show that in fact $\xi_0(x,y) \in L^2(\Omega;\, W)$. Take in (3.1.6) $\phi(x) = \phi^\varepsilon(x) = \varepsilon\,\Phi\left(x, \frac{x}{\varepsilon}\right)$ for any $\Phi(x,y) \in \left[C_0^\infty\left(\Omega; C_{\#}^\infty(Q)\right)\right]^n$. Passing then to the limit in (3.1.6) we notice, via (3.1.7), that the limit of each term but the first one on the left hand-side of (3.1.6) is zero, and therefore

$$\lim_{\varepsilon\to 0}\int_\Omega a^{(1)}\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon(x)\,\cdot\,\varepsilon\nabla\Phi\left(x,\frac{x}{\varepsilon}\right)\,dx$$
$$= \int_\Omega\int_Q \left(a^{(1)}(y)\right)^{1/2}\xi_0(x,y)\,\cdot\,\nabla_y\Phi(x,y)\,dx\,dy \;=\; 0.$$

The density of $\Phi(x,y)$ implies then that, for a.e. x,

$$\operatorname{div}_y\Big(\left(a^{(1)}(y)\right)^{1/2}\xi_0(x,y)\Big) \;=\; 0 \quad \text{in } H^{-1}_{\#}(Q).$$

This yields $\xi_0(x,y) \in L^2(\Omega;\, W)$, see (3.1.12).

Furthermore, a priori estimates (3.1.7) and the weak two-scale compactness theorem tell us that u^ε and $\varepsilon\nabla u^\varepsilon$ two-scale converge, up to some discarded subsequence, to u_0 and ν_0 in $[L^2(\Omega\times Q)]^n$ and $[L^2(\Omega\times Q)]^{n\times d}$ respectively. By definition of two-scale convergence, for fixed $\phi \in C_0^\infty(\Omega)$ and $\psi \in C_{\#}^\infty(Q)$ we have

$$\int_\Omega u^\varepsilon(x)\phi(x)\psi(x/\varepsilon)\ \mathrm{d}x \longrightarrow \int_\Omega\int_Q u_0(x,y)\phi(x)\psi(y)\ \mathrm{d}y\mathrm{d}x$$
$$\int_\Omega \varepsilon\nabla u^\varepsilon(x)\phi(x)\psi(x/\varepsilon)\ \mathrm{d}x \longrightarrow \int_\Omega\int_Q \nu_0(x,y)\phi(x)\psi(y)\ \mathrm{d}y\mathrm{d}x.$$

Noticing that, via integration by parts,

$$\int_\Omega \varepsilon\nabla u^\varepsilon(x)\phi(x)\psi(x/\varepsilon)\ \mathrm{d}x = -\varepsilon\int_\Omega u^\varepsilon(x)\operatorname{div}_x(\phi(x))\psi(x/\varepsilon)\ \mathrm{d}x$$
$$-\int_\Omega u^\varepsilon(x)\phi(x)\operatorname{div}_y(\psi(x/\varepsilon))\ \mathrm{d}x \longrightarrow -\int_\Omega\int_Q u_0(x,y)[\phi(x)\operatorname{div}_y(\psi(y))]\ \mathrm{d}y\mathrm{d}x,$$

we find

$$\int_\Omega \int_Q \nu_0(x,y)\phi(x)\psi(y)\ \mathrm{d}y\mathrm{d}x = -\int_\Omega \int_Q u_0(x,y)[\phi(x)\mathrm{div}_y(\psi(y))]\ \mathrm{d}y\mathrm{d}x,$$

i.e., via the fact $\phi(x)\psi(y)$ span $L^2(\Omega\times Q)$, $u_0(x,y) \in \left[L^2(\Omega; H^1_\#(Q))\right]^n$ with weak derivative $\nabla_y u_0(x,y) = \nu_0(x,y)$.

It remains to show that $u_0(x,y) \in L^2\left(\Omega;\ V\right)$. For any $\psi(x,y) \in \left[C_0^\infty\left(\Omega; C^\infty_\#(Q)\right)\right]^{n\times d}$,

$$\lim_{\varepsilon\to 0}\int_\Omega \left(a^{(1)}\left(\frac{x}{\varepsilon}\right)\right)^{1/2} \varepsilon\nabla u^\varepsilon(x)\cdot\psi\left(x,\frac{x}{\varepsilon}\right)\,dx \\ = \int_\Omega\int_Q \left(a^{(1)}(y)\right)^{1/2}\nabla_y u_0(x,y)\cdot\psi(x,y)\,dx\,dy, \quad (3.1.13)$$

where we have used (3.1.9) with $u_0(x,y) \in \left[L^2\left(\Omega;\ H^1_\#(Q)\right)\right]^n$ and the assumption of boundedness of $a^{(1)}$. On the other hand, (3.1.7) ensures that

$$\left\|\left(a^{(1)}\left(\frac{x}{\varepsilon}\right)\right)^{1/2}\varepsilon\nabla u^\varepsilon(x)\right\|_2 \to 0,$$

and hence the limit in (3.1.13) is zero. This implies for the right hand side of (3.1.13),

$$\int_\Omega\int_Q \left(a^{(1)}(y)\right)^{1/2}\nabla_y u_0(x,y)\cdot\psi(x,y)\,dx\,dy \;=\; 0, \quad \forall\psi\in\left[C_0^\infty\left(\Omega; C^\infty_\#(Q)\right)\right]^{n\times d}.$$

By density of ψ, this gives

$$\left(a^{(1)}(y)\right)^{1/2}\nabla_y u_0(x,y) \;=\; 0 \;\text{ for a.e. }\; x, \qquad (3.1.14)$$

and therefore, pre-multiplying (3.1.14) by $\left(a^{(1)}(y)\right)^{1/2}$, yields $u_0(x,y)\in L^2\left(\Omega;\ V\right)$, cf (3.1.11). □

An important step in the homogenisation process is to find how ξ_0 is related to u_0. We establish that the following relation holds.

Lemma 3.1.2. *Let $u_0(x,y)$ and $\xi_0(x,y)$ be as in Lemma 3.1.1. Then the follow-*

ing integral identity holds:

$$\forall \Psi(x,y) \in C^\infty(\Omega;\, W), \qquad \int_\Omega \int_Q \xi_0(x,y) \,\cdot\, \Psi(x,y)\ \mathrm{d}y \mathrm{d}x = \\ - \int_\Omega \int_Q u_0(x,y) \cdot \mathrm{div}_x \left(\left(a^{(1)}(y)\right)^{1/2} \Psi(x,y) \right) \mathrm{d}y \mathrm{d}x. \quad (3.1.15)$$

Proof. Let $\Psi(x,y) \in C^\infty(\Omega;\, W)$ where W is defined by (3.1.12). Then, by (3.1.10),

$$\lim_{\varepsilon \to 0} \int_\Omega \left(a^{(1)} \left(\frac{x}{\varepsilon} \right) \right)^{1/2} \nabla u^\varepsilon(x) \,\cdot\, \Psi\left(x, \frac{x}{\varepsilon}\right)\, dx \;=\; \int_\Omega \int_Q \xi_0(x,y) \,\cdot\, \Psi(x,y)\, dx\, dy. \quad (3.1.16)$$

On the other hand, integrating by parts and using $u^\varepsilon \in [H^1_0(\Omega)]^n$ and (3.1.12), gives

$$\int_\Omega \left(a^{(1)} \left(\frac{x}{\varepsilon} \right) \right)^{1/2} \nabla u^\varepsilon(x) \,\cdot\, \Psi\left(x, \frac{x}{\varepsilon}\right)\, dx \;= \\ - \int_\Omega u^\varepsilon(x) \,\cdot\, \mathrm{div}_x \left(\left(a^{(1)}(y)\right)^{1/2} \Psi(x,y) \right)\Big|_{y=x/\varepsilon} dx. \quad (3.1.17)$$

Passing to the limit and using (3.1.8) then yields

$$\lim_{\varepsilon \to 0} \int_\Omega \left(a^{(1)} \left(\frac{x}{\varepsilon} \right) \right)^{1/2} \nabla u^\varepsilon(x) \,\cdot\, \Psi\left(x, \frac{x}{\varepsilon}\right)\, dx \;= \\ - \int_\Omega \int_Q u_0(x,y) \,\cdot\, \mathrm{div}_x \left(\left(a^{(1)}(y)\right)^{1/2} \Psi(x,y) \right)\, dx\, dy. \quad (3.1.18)$$

Comparing (3.1.16) and (3.1.18) results in identity (3.1.15). □

Notice that the test function $\Psi(x,y)$ in the above lemma does not have to vanish when $x \in \partial\Omega$. Formally, if we use integration by parts on the right hand side of (3.1.15) we find

$$\int_\Omega \int_Q [\xi_0(x,y) - \left(a^{(1)}(y)\right)^{1/2} \nabla_x u_0(x,y)] \cdot \Psi(x,y)\ \mathrm{d}y \mathrm{d}x = 0, \quad (3.1.19)$$

i.e. $\xi_0(x,y) - \left(a^{(1)}(y)\right)^{1/2} \nabla_x u_0(x,y) \perp \Psi(x,y)$ in $[L^2(\Omega \times Q)]^{n \times d}$. However it should be mentioned here that there is generally no reason why $\nabla_x u_0 \in$

$[L^2(\Omega \times Q)]^{n\times d}$. In the classical and high contrast regime, (3.1.19) is known to be valid, see Section 3.2, and this, along with Weyl's decomposition, implies that

$$\xi_0(x,y) = \chi_1(y)\left(a^{(1)}(y)\right)^{1/2}\left[\nabla_x u_0(x,y) + \nabla_y u_1(x,y)\right],$$

for some $u_1(x,y)$ solving the cell problem

$$-\mathrm{div}_y\Big(\chi_1(y)a^{(1)}(y)\left[\nabla_y u_1(x,y) + \nabla_x u_0(x,y)\right]\Big) = 0.$$

For the general partially degenerate case, $a^{(1)} \geq 0$, to prove the generalised Weyl's decomposition, see Lemma 3.1.3, it is sufficient to impose the following key assumption on $a^{(1)}$:

Key assumption on the degeneracy: *There exists a constant $C > 0$ such that for all $v \in \left[H^1_\#(Q)\right]^n$ there exists $v_1 \in V$ with*

$$\|v - v_1\|_{\left[H^1_\#(Q)\right]^n} \leq C\,\left\|a^{(1)}(y)\nabla_y v\right\|_2. \tag{3.1.20}$$

The condition (3.1.20) can obviously be equivalently re-written as

$$\|P_{V^\perp}v\|_{\left[H^1_\#(Q)\right]^n} \leq C\,\left\|a^{(1)}(y)\nabla_y v\right\|_2, \tag{3.1.21}$$

where $P_{V^\perp}$ is the orthogonal projector in $\left[H^1_\#(Q)\right]^n$ on $V^\perp$, the orthogonal complement to V. (The equivalence of (3.1.20) and (3.1.21) immediately follows by noticing that $v_1 = P_V v$, where P_V denotes the orthogonal projector on V, is the best choice of v_1 for (3.1.20).) The assumption (3.1.20) clearly does not depend on the choice of an equivalent norm in $H^1_\#$. We remark here that the assumption (3.1.20) does hold for most of the particular cases, including: the classical homogenisation and 'fully' high contrast homogenisation cases, see Section 3.2, and the cases of 'genuinely' partial degeneracies considered in Chapters 4 and 5 of this thesis. In general, (3.1.20) has to be checked by separate means for particular examples. The next lemma establishes a generalisation of Weyl's decomposition.

Lemma 3.1.3 (Generalised Weyl's decomposition.)**.** *Let $a^{(1)}$ satisfy* (3.1.20)*, and* *let* $\eta \in \left[L^2_\#(Q)\right]^{n\times d}$. *Suppose* η *is orthogonal in* $\left[L^2_\#(Q)\right]^{n\times d}$ *to* W, $\eta \in W^\perp$, *i.e.*

$$(\,\eta\,,\,\psi\,)_2 \,:=\, \int_Q \eta_{ij}(y)\psi_{ij}(y)dy \,=\, 0, \quad \forall\,\psi\,\in\,W. \tag{3.1.22}$$

Then there exists $u_1 \in \left[H^1_\#(Q)\right]^n$ *such that*

$$\eta(y) \,=\, \left(a^{(1)}(y)\right)^{1/2}\nabla_y u_1(y). \tag{3.1.23}$$

Such a u_1 *is determined uniquely up to any function from* V, *in particular is unique in* $V^\perp$.

The proof of Lemma 3.1.3 will follow shortly. The fact that the Generalised Weyl's decomposition holds is closely related to the existence of solutions to the following general degenerate 'unit cell' problem: find $v \in [H^1_\#(Q)]^n$ such that

$$-\mathrm{div}_y\Big(a^{(1)}\nabla_y v(y)\Big) = F \tag{3.1.24}$$

for given $F \in \left[H^{-1}_\#(Q)\right]^n$ (here $\left[H^{-1}_\#(Q)\right]^n$ is the dual space of $[H^1_\#(Q)]^n$, i.e. the space of bounded linear functionals on $[H^1_\#(Q)]^n$). The weak formulation of (3.1.24) is: find $v \in \left[H^1_\#(Q)\right]^n$ such that

$$\int_Q a^{(1)}(y)\nabla_y v(y)\cdot\nabla_y w(y)\,dy \,=\, \langle\,F\,,\,w\,\rangle, \quad \forall\,w \in \left[H^1_\#(Q)\right]^n. \tag{3.1.25}$$

The key assumption (3.1.20), equivalently (3.1.21), is sufficient for the existence of solutions to (3.1.25), provided F is as follows.

Lemma 3.1.4.
(i) The problem (3.1.25) *is solvable in* $\left[H^1_\#(Q)\right]^n$ *if and only if*

$$\langle\,F\,,\,w\,\rangle \,=\, 0, \quad \forall\,w \in V. \tag{3.1.26}$$

When (3.1.26) *does hold, the problem* (3.1.25) *is uniquely solvable in* $V^\perp$.
(ii) For any solution v *and any* $v_1 \in V$, $v + v_1$ *is also a solution. Conversely, any two solutions can only differ by a* $v_1 \in V$.

Proof. (i) Let v be a solution of (3.1.25) and let $w \in V$. Then, using the symmetry of $a^{(1)}$ and (3.1.11),

$$\langle F\,,\,w\rangle \;=\; \int_Q a^{(1)}(y)\nabla_y v(y)\cdot\nabla_y w(y)\,dy \;=\; \int_Q \nabla_y v(y)\cdot a^{(1)}(y)\nabla_y w(y)\,dy \;=\; 0 \tag{3.1.27}$$

yielding (3.1.26). Conversely let (3.1.26) hold, and seek $v \in \left[H^1_\#(Q)\right]^n$ solving (3.1.25). By (3.1.27), the identity (3.1.25) is automatically held for all w in V, therefore it is sufficient to verify it for all $w \in V^\perp$. Seek v also in $V^\perp$. Show that then, in the Hilbert space $H := V^\perp$ with the inherited $\left[H^1_\#(Q)\right]^n$ norm $\|\cdot\|_H$, the problem (3.1.25) satisfies the conditions of the Lax-Milgram lemma, see Lemma C.0.3. Namely, first the bilinear form

$$B[v,w] \;:=\; \int_Q a^{(1)}(y)\nabla_y v(y)\cdot\nabla_y w(y)\,dy$$

is shown to be bounded in H, i.e. with some $C>0$,

$$\Big|B[\,v\,,\,w\,]\Big| \;\le\; C\,\|v\|_H\,\|w\|_H, \quad \forall v,\,w\;\in\;H.$$

This follows from (3.1.3). Choosing as the inner product

$$(v,w)_{H^1} \;=\; \left(\int_Q v\right)\left(\int_Q w\right) \;+\; \int_Q \nabla v\cdot\nabla w,$$

we will now show that the form B is coercive, i.e. for some $\nu>0$,

$$B[v,v] \;\ge\; \nu\,\|v\|^2_{H^1}, \quad \forall v\in V^\perp.$$

We have, by the boundedness of $\left(a^{(1)}\right)^{1/2}$,

$$B[v,v] := \int_Q a^{(1)}(y)\nabla_y v(y)\cdot\nabla_y v(y)\,dy = \left\|\left(a^{(1)}(y)\right)^{1/2}\nabla_y v\right\|_2^2 \ge$$

$$C\,\left\|\,a^{(1)}(y)\nabla_y v\,\right\|_2^2 \;\ge\; \nu\,\|v\|^2_H.$$

In the last two inequalities we have used, the boundedness of $\left(a^{(1)}\right)^{1/2}$, (3.1.21) and the fact

$$V^\perp \subset \left\{u\in[H^1(Q)]^n : \int_Q u = 0\right\}.$$

.Therefore, by the Lax-Milgram lemma, there exists a unique solution to the problem

$$v \in V^{\perp}: \quad B[v,w] \,=\, \langle\, F\,,\, w\,\rangle, \quad \forall w \,\in\, V^{\perp},$$

and hence to (3.1.25).

(ii) If v solves (3.1.25) and $v_1 \in V$ then $a^{(1)}(y)\nabla_y v_1(y) = 0$ and hence $v + v_1$ also solves (3.1.25).

Assuming further $v^{(1)}$ and $v^{(2)}$ both solve (3.1.25), set $v = v^{(1)} - v^{(2)}$ solving hence (3.1.25) with $F = 0$, and then set $w = v$. As a result,

$$0 \,=\, \int_Q a^{(1)}(y)\nabla_y v(y) \cdot \nabla_y v(y)\, dy \,=\, \left\| \left(a^{(1)}(y)\right)^{1/2} \nabla_y v \right\|_2^2,$$

implying $\left(a^{(1)}\right)^{1/2} \nabla_y v \,=\, 0$ and hence $a^{(1)}\nabla_y v \,=\, 0$, i.e. $v \in V$. □

Proof of Lemma 3.1.3. Let η satisfying (3.1.22) be given, and seek u_1 such that (3.1.23) holds. Left-multiply (3.1.23) by $\left(a^{(1)}(y)\right)^{1/2}$ and take the divergence of both sides. As a result,

$$-\ \mathrm{div}_y \left(a^{(1)}(y)\nabla_y u_1\right) \,=\, -\ \mathrm{div}_y \left(\left(a^{(1)}(y)\right)^{1/2} \eta \right) \,=:\, F. \qquad (3.1.28)$$

Check that the above defined $F \in \left[H^{-1}_{\#}(Q)\right]^n$ satisfies the condition (3.1.26). For $w \in V$,

$$\langle\, F\,,\, w\,\rangle \,=\, -\left\langle\ \mathrm{div}_y \left(\left(a^{(1)}(y)\right)^{1/2} \eta \right)\,,\, w \right\rangle \,=$$

$$\int_Q \left(a^{(1)}(y)\right)^{1/2} \eta \cdot \nabla w(y)\, dy \,=\, \int_Q \eta \cdot \left(a^{(1)}(y)\right)^{1/2} \nabla w(y)\, dy. \qquad (3.1.29)$$

Since $w \in V$ it follows that $a^{(1)}(y)\nabla w(y) \,=\, 0$ for a.e. y, and hence (for a.e. y) $\left(a^{(1)}(y)\right)^{1/2} \nabla w(y) \,=\, 0$. (Since for any $\xi \in \mathbb{R}^{n\times d}$, $\left(a^{(1)}(y)\right)^{1/2} \xi \,=\, 0$ if and only if $a^{(1)}(y)\xi = 0$ by the symmetry of non-negative $a^{(1)}$.) This implies that the expression in (3.1.29) vanishes, and hence $\langle\, F\,,\, w\,\rangle \,=\, 0$, i.e. (3.1.26) holds.

Then, by Lemma 3.1.4, there exists a unique $u_1 \in V^{\perp}$ such that (3.1.28) holds. Verify that such a u_1 satisfies (3.1.23). We have

$$\left\| \eta(y) \,-\, \left(a^{(1)}(y)\right)^{1/2} \nabla_y u_1(y) \right\|_2^2 \,=\, \left(\eta(y)\,,\ \ \eta(y) \,-\, \left(a^{(1)}(y)\right)^{1/2} \nabla_y u_1(y)\right)_2 \,-$$

$$\left(\left(a^{(1)}(y)\right)^{1/2}\nabla_y u_1(y)\,,\quad \eta(y)\,-\,\left(a^{(1)}(y)\right)^{1/2}\nabla_y u_1(y)\right)_2 \;=:\; S_1\,+\,S_2. \qquad (3.1.30)$$

Now, it follows from (3.1.28) that $\psi(y) := \eta(y) \,-\, \left(a^{(1)}(y)\right)^{1/2}\nabla_y u_1(y) \in W$ (see (3.1.12)), and hence, by the assumption (3.1.22) of the lemma, $S_1 = 0$. On the other hand,

$$S_2 := \int_Q \left(a^{(1)}(y)\right)^{1/2}\nabla_y u_1(y)\,\cdot\,\psi(y)\,dy \,=$$

$$\int_Q \nabla_y u_1(y)\,\cdot\,\left(a^{(1)}(y)\right)^{1/2}\psi(y)\,dy \;=:\; -\left\langle\,\mathrm{div}\left(\left(a^{(1)}(y)\right)^{1/2}\psi(y)\right)\,,\,u^{(1)}\right\rangle \,=0$$

by (3.1.28). Hence (3.1.30) yields $\left\|\eta(y)\,-\,\left(a^{(1)}(y)\right)^{1/2}\nabla_y u_1(y)\right\|_2 = 0$ implying (3.1.23).

The above construction also ensures that u_1 is determined uniquely up to any function from V, in particular is unique in $V^\perp$. □

Note that, under the key assumption (3.1.20), equivalently (3.1.21), if we are able to characterise the generalised flux ξ_0 in terms of the limit function u^0 we shall be able to pass to the two-scale limit in (3.1.6). We notice that, by Lemmas 3.1.1 and 3.1.2, u_0 belongs to the following linear subspace U of $L^2(\Omega;V)$

$$\begin{aligned} U \;:=\; \Bigg\{\, u(x,y)\in L^2\,(\Omega;\,V)\;\Bigg|\;\exists\,\xi(x,y)\in L^2(\Omega;W)\text{ such that,}\\ \forall\,\Psi(x,y)\in C^\infty(\Omega;\,W),\qquad \int_\Omega\int_Q \xi(x,y)\,\cdot\,\Psi(x,y)\,dx\,dy = \\ -\int_\Omega\int_Q u(x,y)\,\cdot\,\mathrm{div}_x\left(\left(a^{(1)}(y)\right)^{1/2}\Psi(x,y)\right)\,dx\,dy\Bigg\}. \end{aligned} \qquad (3.1.31)$$

Note that for a given $u\in U$, the associated $\xi\in L^2(\Omega;W)$ in (3.1.31) is unique by the density of $C^\infty(\Omega;W)$ in $L^2(\Omega;W)$. We can hence define the linear operator $T : U \to L^2(\Omega;W)$ by $Tu = \xi$. We equip U with the inner product

$$(u,v)_U := \int_\Omega\int_Q Tu\cdot Tv + a^{(0)}(y)\nabla_y u\cdot\nabla_y v\;\mathrm{d}y\mathrm{d}x + \alpha\int_\Omega\int_Q \rho(y)u\cdot v\;\mathrm{d}y\mathrm{d}x, \qquad (3.1.32)$$

and the norm induced by the inner product

$$\|u\|_U^2 = (u,u)_U. \tag{3.1.33}$$

Lemma 3.1.5.

(i) $T : U \to L^2(\Omega; W)$ defined by $Tu \mapsto \xi$ is bounded.

(ii) U is a Hilbert space.

Proof. To show T is bounded note that,

$$\begin{aligned}\|Tu\|^2_{L^2(\Omega\times Q)} \le \int_\Omega\int_Q |Tu|^2 \, \mathrm{d}y\mathrm{d}x + \int_\Omega\int_Q a^{(0)}(y)\nabla_y u\cdot\nabla_y u \, \mathrm{d}y\mathrm{d}x \\ +\alpha\int_\Omega\int_Q \rho(y) u\cdot u \, \mathrm{d}y\mathrm{d}x = \|u\|_U^2.\end{aligned} \tag{3.1.34}$$

We will now show that U is in fact a Hilbert space equipped with the inner product (3.1.32): Let u_j, $j=1,2,...$, be a Cauchy sequence in U, i.e. $\|u_j - u_k\|_U \to 0$ as $j,k\to\infty$. Let $\xi_j := Tu_j$. Then, according to (3.1.34) and (3.1.5), $\|u_j-u_k\|_{L^2(\Omega;V)} \to 0$ and $\|\xi_j-\xi_k\|_{L^2(\Omega;W)}\to 0$. Since both $L^2(\Omega;V)$ and $L^2(\Omega;W)$ are complete, there exist $\tilde u\in L^2(\Omega;V)$ and $\tilde\xi\in L^2(\Omega;W)$ such that, respectively, $u_j\to\tilde u$ in $L^2(\Omega;V)$ and $\xi_j\to\tilde\xi$ in $L^2(\Omega;W)$. Taking then arbitrary $\Psi(x,y)\in C^\infty(\Omega;W)$, since $u_j\in U$,

$$\begin{aligned}&\int_\Omega\int_Q \xi_j(x,y)\cdot\Psi(x,y)\,\mathrm{d}y\mathrm{d}x \\ &\qquad = -\int_\Omega\int_Q u_j(x,y)\cdot \mathrm{div}_x\Big(\big(a^{(1)}(y)\big)^{1/2}\,\Psi(x,y)\Big)\,\mathrm{d}y\mathrm{d}x\end{aligned}$$

Passing to the limit as $j\to\infty$ in both the left hand side and the right hand side of the above identity shows that it also holds for $\tilde\xi$ and $\tilde u$. Therefore $\tilde u\in U$, $\tilde\xi = T\tilde u$, and $\|u_j-\tilde u\|_U\to 0$. Hence U is complete. □

If we choose test functions for (3.1.6) of the form $\phi^\varepsilon = \phi_0(x, x/\varepsilon)$ for ϕ_0 that behave like u_0, that is ϕ_0 belongs to the space U, passing to the two-scale limit as $\varepsilon\to 0$ in (3.1.6) will give the equation that characterises u_0, i.e. the two-scale homogenised limit equation corresponding (3.1.6). The following limit theorem is stated and proved for strictly star-shaped domains. This is a technical restriction which could be relaxed.

Definition 3.1.6. We call Ω a strictly star shaped domain if:

$$\forall \delta > 0 \qquad \operatorname{dist}((1-\delta)\,\Omega, \partial\Omega) > 0.$$

Theorem 3.1.7. *Let Ω be strictly star shaped or $\Omega = \mathbb{R}^d$. Let f^ε weakly (strongly) two-scale converge to $f(x,y)$. Then u^ε, the solution to* (3.1.6)*, weakly (strongly) two-scale converges to $u_0 \in U$, where u_0 is the unique solution of the following two-scale homogenised equation.*

$$\int_\Omega \int_Q T u_0 \cdot T\phi_0 + a^{(0)}(y)\nabla_y u_0 \cdot \nabla_y \phi_0 \; \mathrm{d}y\mathrm{d}x \;+ \alpha \int_\Omega \int_Q \rho(y) u_0 \cdot \phi_0 \; \mathrm{d}y\mathrm{d}x = \int_\Omega \int_Q f \cdot \phi_0 \; \mathrm{d}y\mathrm{d}x, \quad \forall \phi_0 \in U. \quad (3.1.35)$$

Proof. The left hand side of (3.1.35) is precisely $(u, \phi_0)_U$, see (3.1.32), while the right hand side of (3.1.35) defines a bounded linear functional on U. Therefore, by the Riesz representation theorem, problem (3.1.35) is well posed, i.e. there exists a unique solution to (3.1.35).

For fixed $\phi_0(x,y) \in C_0^\infty(\Omega; V)$, let $\phi_1(x,y)$ be a solution to

$$-\operatorname{div}_y[(a^{(1)})^{1/2}\nabla_y \phi_1(x,y)] = \operatorname{div}_y[(a^{(1)})^{1/2}(y)\nabla_x \phi_0(x,y)].$$

Then for $\eta(x,y) := (a^{(1)})^{1/2}(y)[\nabla_y \phi_1 + \nabla_x \phi_0]$ using integration by parts and (3.1.12) we find that

$$\forall \Psi(x,y) \in C^\infty(\Omega;\, W), \qquad \int_\Omega \int_Q \eta(x,y) \,\cdot\, \Psi(x,y) \; \mathrm{d}y\mathrm{d}x = - \int_\Omega \int_Q \phi_0(x,y) \cdot \operatorname{div}_x \Big(\big(a^{(1)}(y)\big)^{1/2} \Psi(x,y) \Big) \; \mathrm{d}y\mathrm{d}x,$$

i.e., via (3.1.31), $\phi_0 \in U$ with $T\phi_0 = \eta$. Taking $\phi^\varepsilon(x) = \phi_0(x, x/\varepsilon) + \varepsilon\phi_1(x, x/\varepsilon)$

as test functions in (3.1.6) gives, using (3.1.11),

$$\int_\Omega a^{(1)}(x/\varepsilon)\nabla u^\varepsilon\cdot[\nabla_x\phi_0+\varepsilon\nabla_x\phi_1+\nabla_y\phi_1]+$$
$$+\int_\Omega \varepsilon^2 a^{(0)}(x/\varepsilon)\nabla u^\varepsilon\cdot[\nabla_x\phi_0+\varepsilon^{-1}\nabla_y\phi_0+\varepsilon\nabla_x\phi_0+\nabla_y\phi_1]+$$
$$+\alpha\int_\Omega \rho^\varepsilon u^\varepsilon\cdot[\phi_0+\varepsilon\phi_1]=\int_\Omega f^\varepsilon\cdot[\phi_0+\varepsilon\phi_1].$$

Using the symmetry of $a^{(l)}$, $l=0,1$, the a priori bounds (3.1.7) and Lemma 3.1.1 we pass to the two-scale limit in the above equation, yielding

$$\int_\Omega\int_Q \xi_0(x,y)\cdot(a^{(1)})^{1/2}(y)[\nabla_x\phi(x,y)+\nabla_y\phi_1(x,y)]\;\mathrm{d}y\mathrm{d}x+$$
$$+\int_\Omega\int_Q a^{(0)}(y)\nabla_y u_0(x,y)\cdot\nabla_y\phi_0(x,y)\;\mathrm{d}y\mathrm{d}x+\alpha\int_\Omega\int_Q\rho(y)u_0(x,y)\cdot\phi_0(x,y)\;\mathrm{d}y\mathrm{d}x$$
$$=\int_\Omega\int_Q f(x,y)\cdot\phi_0(x,y)\;\mathrm{d}y\mathrm{d}x. \quad (3.1.36)$$

Since $\xi_0=Tu_0$ and $T\phi_0=(a^{(1)})^{1/2}(y)[\nabla_x\phi(x,y)+\nabla_y\phi_1(x,y)]$, (3.1.36) reads

$$\int_\Omega\int_Q Tu_0\cdot T\phi_0\;\mathrm{d}y\mathrm{d}x+\int_\Omega\int_Q a^{(0)}(y)\nabla_y u_0\cdot\nabla_y\phi_0\;\mathrm{d}y\mathrm{d}x+$$
$$+\alpha\int_\Omega\int_Q\rho(y)u_0\cdot\phi_0\;\mathrm{d}y\mathrm{d}x=\int_\Omega\int_Q f\cdot\phi_0\;\mathrm{d}y\mathrm{d}x,\quad\forall\phi_0\in C_0^\infty(\Omega;V).$$

Therefore, to prove (3.1.35) it is sufficient to show that $C_0^\infty(\Omega;V)$ is dense in U with respect to the norm (3.1.33). This property will be proved for star-shaped bounded domains Ω, or $\Omega=\mathbb{R}^d$.

Let Ω be a domain, strictly star-shaped with respect to origin O. Fix $u(x,y)\in U$, let $\xi(x,y)=Tu(x,y)\in L^2(\Omega;W)$, and regard both u and ξ as functions on the whole $\mathbb{R}^d$ in x by extending them outside Ω by zero. We aim at constructing a sequence $u_\delta\in C_0^\infty(\Omega;V)$ such that $u_\delta\to u$ in U as $\delta\to 0$.

To this end, following [21], for any small $\delta>0$, let $\Omega_\delta:=(1-\delta)\Omega$ and denote $d(\delta):=\mathrm{dist}\,(\Omega_\delta,\partial\Omega)>0$. Let $\hat{u}_\delta(x,y):=u(x/(1-\delta),y)$. Obviously, the support of $\hat{u}_\delta$ is contained in $\overline{\Omega_\delta}\subset\Omega$. Select $\epsilon(\delta)=d(\delta)/2>0$ and let $\zeta_\epsilon(x)$ be a standard mollifying function: $\zeta_\epsilon(x)=\epsilon^{-d}\zeta(x/\epsilon)$, where $\zeta(x)\in C_0^\infty(\mathbb{R}^d)$, $\zeta(-x)=\zeta(x)$, supp $\zeta(x) \subset B(0,1)$ and $\int_{\mathbb{R}^d} \zeta(x)dx = 1$. Consider the x-smoothed function

$$u_\delta(x,y) \;:=\; \zeta_\epsilon * \hat{u}_\delta(x,y) \;:=\; \int_{\mathbb{R}^d} \zeta_\epsilon(x-x')\hat{u}_\delta(x',y)dx'.$$

Obviously, by construction, $u_\delta(x,y) \in C_0^\infty(\Omega;V) \subset U$.

We argue that $u_\delta \to u$ in U as $\delta \to 0$. According to (3.1.34) it suffices to show that $u_\delta \to u$ in $L^2(\Omega;V)$ and $Tu_\delta \to Tu$ in $L^2(\Omega;W)$.

The former assertion immediately follows from the fact that $\hat{u}_\delta \to u$ in $L^2(\Omega;V)$, cf. e.g. [18], and from $\|u_\delta - \hat{u}_\delta\|_{L^2(\Omega;V)} \to 0$ (trivially established via e.g. changing variables $\hat{x} = x/(1-\delta)$, noticing that $\epsilon \to 0$ as $\delta \to 0$ and using the properties of the mollifications, cf. [18]).

To prove that $Tu_\delta \to Tu$, choose $\Psi(x,y) \in C^\infty(\Omega;V)$. Then, for the right hand side of (3.1.31) with u replaced by u_δ,

$$I(\delta) \;:=\; -\int_\Omega\int_Q u_\delta(x,y) \,\cdot\, \mathrm{div}_x\left(\left(a^{(1)}(y)\right)^{1/2}\Psi(x,y)\right)dy\,dx \;=$$

$$-\int_{\Omega_{\delta/2}}\int_Q\left[\int_{\Omega_\delta}\zeta_\epsilon(x-x')\hat{u}_\delta(x',y)dx'\right]\cdot\,\mathrm{div}_x\left(\left(a^{(1)}(y)\right)^{1/2}\Psi(x,y)\right)dy\,dx \;=$$

$$-\int_{\Omega_\delta}\int_Q \hat{u}_\delta(x',y)\,\cdot\, I_\epsilon(x',y)\,dydx', \tag{3.1.37}$$

where

$$I_\epsilon(x',y) \;:=\; \int_\Omega \zeta_\epsilon(x-x')\mathrm{div}_x\left(\left(a^{(1)}(y)\right)^{1/2}\Psi(x,y)\right)dx,$$

having interchanged above the order of integration. Notice that for $x' \in \Omega_\delta$ the integrand in $I_\epsilon(x',y)$, is smooth and compactly supported in Ω in x. Hence, via integration by parts and straightforward manipulation,

$$I_\epsilon(x',y) \;=\; \mathrm{div}_{x'}\left(\left(a^{(1)}(y)\right)^{1/2}\hat{\Psi}_\delta(x',y)\right), \tag{3.1.38}$$

where

$$\hat{\Psi}_\delta(x',y) \;:=\; \zeta_\epsilon * \Psi(x',y) := \int_\Omega \zeta_\epsilon(x''-x')\Psi(x'',y)dx'' \;\in C^\infty(\Omega_\delta;W). \tag{3.1.39}$$

Changing in (3.1.37)–(3.1.38) the integration variable ($x = x'/(1-\delta)$), and introducing $\Psi_\delta(x,y) := (1-\delta)^{d-1}\hat{\Psi}_\delta\left((1-\delta)x, y\right) \in C^\infty(\Omega; W)$, results in

$$I(\delta) \;=\; -\int_\Omega\int_Q u(x,y)\,\cdot\,\mathrm{div}_x\Big(\left(a^{(1)}(y)\right)^{1/2}\Psi_\delta(x,y)\Big)\,dy\,dx,$$

which reproduces the right hand side of (3.1.31) for Ψ replaced by Ψ_δ. Hence, applying (3.1.31) to $u \in U$ and $\Psi_\delta \in C^\infty(\Omega_\delta; W)$ results in

$$I(\delta) \;=\; \int_\Omega\int_Q \xi(x,y)\,\cdot\,\Psi_\delta(x,y)\,dy\,dx \;=\; \int_\Omega\int_Q \xi_\delta(x,y)\,\cdot\,\Psi(x,y)\,dy\,dx, \quad (3.1.40)$$

where, via (3.1.39), and a further change of integration variables,

$$\xi_\delta(x,y) \;:=\; (1-\delta)^{-1}\int_{\Omega_\delta}\zeta_\epsilon(x-x')\xi\left(x'/(1-\delta), y\right)dx'. \quad (3.1.41)$$

By the uniqueness of ξ in (3.1.31) for u replaced by $u_\delta \in U$, $Tu_\delta = \xi_\delta$. It is now straightforward to check for ξ_δ, as given by (3.1.41), $\xi_\delta \to \xi = Tu$ in $L^2(\Omega; W)$ as $\delta \to 0$. Therefore $Tu_\delta \to Tu$ in $L^2(\Omega; W)$ as $\delta \to 0$, which completes the proof.

The proof in the case of $\Omega = \mathbb{R}^d$ is similar to the above with $\hat{u}_\delta$ replaced by multiplying u by a suitable family of cut-off functions. □

Remark. We shall see in the next section that for the classical and high contrast regimes we can improve the regularity in x on the space U and as such the generalised homogenised matrix form is

$$\int_\Omega\int_Q Tu\cdot Tv\;\mathrm{d}y\mathrm{d}x = \int_\Omega\int_Q a^{\mathrm{hom}}(y)\nabla_x u\cdot\nabla_x v\;\mathrm{d}y\mathrm{d}x,$$

for an appropriate $a^{\mathrm{hom}}(y)$. The homogenised limit form is in general 'non-local'. This is due to the effect of the degeneracy $a^{(1)}(y)$ which implies that the H^1 regularity of U in x may not hold. The simplest example of such a degenerate $a^{(1)}$ is $a^{(1)}(y) \equiv 0$. The key assumption (3.1.20) does indeed hold for $a^{(1)} \equiv 0$ and it is clear that $U = L^2(\Omega; V)$ with $T \equiv 0$. More complicated examples of such degeneracies can be constructed, from the above observation, by letting $a^{(1)}(y)$ be independent of one or more spacial dimensions, e.g. for $d = 2$, $n = 2$,

$$a^{(1)}(y) = \begin{pmatrix} a(y_1) & 0 \\ 0 & 0 \end{pmatrix}.$$

Then for an appropriate choice of $a(y_1)$ to satisfy the key assumption we see that functions in U will be H^1 in x_1 but L^2 in x_2. In Chapter 5 we will find an example of a non-trivial degeneracy with corresponding $T \equiv 0$, and $U = L^2(\Omega; V)$.

We now define the two-scale limit operator A^0. Denote by H the closure of U in $L^2_\rho(\Omega \times Q)$ with inner product

$$(u, v)_H = \int_\Omega \int_Q \rho(y) u(x, y) \cdot v(x, y) \, \mathrm{d}y\mathrm{d}x.$$

Theorem 3.1.7 implies that

$$\beta(u, v) := \int_\Omega \int_Q Tu \cdot Tv + a^{(0)} \nabla_y u \cdot \nabla_y v \, \mathrm{d}y\mathrm{d}x,$$

defines a bilinear form on the Hilbert space H. The bilinear form β is, clearly, non-negative, and is closed on H since $C_0^\infty(\Omega; V)$ is dense in U. Therefore, it is well known, see e.g. [32], that β defines a non-negative self-adjoint operator A^0, called the Friedrichs extension, by

$$\beta(u, v) = (A^0 u, v)_H, \quad \forall u, v \in \mathcal{D}(A^0),$$

where the domain $\mathcal{D}(A^0) \subset U$ is a dense subset of H. We call A^0 the homogenised limit operator.

Theorem 3.1.7 implies the self-adjoint operator A^ε, the associated Friedrichs extension of problem (3.1.6), converges in the weak two-scale resolvent sense, which implies strong (see Appendix B.1) , to A^0. This, in turn, implies that the limit spectrum, the spectrum of A^0, is contained in the limiting spectrum, see Section B.1 of Appendix B, Proposition B.1.2. That is

$$\text{For given } \lambda^0 \in \sigma(A^0) \text{ there exists } \lambda^\varepsilon \in \sigma(A^\varepsilon) \text{ such that } \lambda^\varepsilon \to \lambda^0 \text{ as } \varepsilon \to 0. \tag{3.1.42}$$

In applications, an additional desired property is the 'spectral compactness':

$$\text{Let } \lambda^\varepsilon \in \sigma(A^\varepsilon) \text{ such that } \lambda^\varepsilon \to \lambda^0. \text{ Then } \lambda^0 \in \sigma(A^0). \tag{3.1.43}$$

Theorem 3.1.7 guarantees property (3.1.42) but does not ensure (3.1.43). In general this may not hold and has to be tested by other means. We shall see that for the partially degenerating PDEs studied in Chapter 4 the spectral compactness result does indeed hold, i.e. properties (i) and (ii) hold. For the problem in Chapter 5 however, the spectral compactness only holds when the limit operator is extended further, to include both periodic and quasi-periodic dependence on the fast variable y.

3.2 Applications to classical and high contrast homogenisation

3.2.1 Classical regime

Considering, in (3.1.2), $n = 1$, $a^{(0)} \equiv 0$ and $a^{(1)} > \nu I$ gives the classical homogenisation problem mentioned in Chapter 2.1. Following the approach to homogenisation outlined in the previous section, we introduce the space of microscopic oscillations

$$V := \left\{ v \in H^1_\#(Q) : a^{(1)} \nabla_y v = 0 \right\}.$$

Since the symmetric tensor $a^{(1)}$ is positive $v \in V$ if, and only if, $\nabla v = 0$ and hence v is constant. By taking in (3.1.20) $v_1 = \langle v \rangle = \int_Q v \, \mathrm{d}y$ the key assumption (3.1.20) takes the form: There exists a constant $C > 0$ such that for all $v \in H^1_\#(Q)$, with $\int_Q v \, \mathrm{d}y = 0$,

$$\|v\|_{H^1_\#} \leq C \|\nabla_y v\|_{L^2}. \tag{3.2.1}$$

This is the well known Poincaré inequality, which holds on the domain Q. Therefore the key assumption does hold. Hence the homogenisation theorem, Theorem 3.1.7, holds. To show that Theorem 3.1.7 corresponds to the classical homogenisation theorem, we need to show the space U is nothing more than $H^1_0(\Omega)$ and

$$\int_\Omega \int_Q T u_0 \cdot T \phi_0 \, \mathrm{d}y \mathrm{d}x = \int_\Omega a^{\mathrm{hom}} \nabla_x u_0 \cdot \nabla_x \phi_0 \, \mathrm{d}x \qquad \forall u_0, \phi_0 \in H^1_0(\Omega) \tag{3.2.2}$$

for a^{hom} given by (2.1.9).

To show $U = H^1_0(\Omega)$, we first note $H^1_0(\Omega) \subset U$ as: for fixed $u \in H^1_0(\Omega)$ define

$$\xi(x, y) := \left(a^{(1)}(y)\right)^{1/2} [\nabla_x u(x) + \nabla_y u_1(x, y)],$$

where $u_1(x,y)$ solves the cell problem

$$-\mathrm{div}_y \left(a^{(1)}(y)[\nabla_y u_1(x,y) + \nabla_x u(x)]\right) = 0.$$

Then, by a simple application of integration by parts, it is clear that the pair (u,ξ) satisfies identity (3.1.15) and therefore $u \in U$. Clearly $Tu = \xi$ and (3.2.2) follows by noticing that $u_1(x,y) = N(y) \cdot \nabla_x u(x)$ where, for $r = 1,\dots,d$, $N_r(y)$ solve the classical cell problem:

$$-\mathrm{div}_y\Big(a^{(1)}(y)\nabla_y N_r(y)\Big) = -\mathrm{div}_y\Big(a^{(1)}(y)e_r\Big). \tag{3.2.3}$$

Let us now show that $H_0^1(\Omega)$ is in fact the whole space U. Assume $H_0^1(\Omega) \neq U$, then there exists $0 \neq w \in (H_0^1)^\perp$, the orthogonal complement of $H_0^1(\Omega)$ in U, such that

$$(w,u)_U = \int_\Omega\int_Q Tw \cdot Tu \; \mathrm{d}y\mathrm{d}x + \alpha \int_\Omega\int_Q wu \; \mathrm{d}y\mathrm{d}x = 0 \quad \forall u \in H_0^1(\Omega). \tag{3.2.4}$$

By the definition of U, (3.1.31) for a.e. $x \in \Omega$, $w(x,\cdot) \in V$ that is $w(x,y) = w(x)$. Therefore

$$\forall\, \Psi(x,y) \in C^\infty(\Omega;\, W), \quad \int_\Omega\int_Q Tw(x,y) \,\cdot\, \Psi(x,y)\, dx\, dy = \\ - \int_\Omega\int_Q w(x) \,\cdot\, \mathrm{div}_x \left(\left(a^{(1)}(y)\right)^{1/2} \Psi(x,y) \right) dx\, dy. \tag{3.2.5}$$

We will now show that (3.2.4)-(3.2.5) imply that $w \in H_0^1$ and therefore imply $w = 0$ giving a contradiction. To prove this it is sufficient to show that there exists a linear map $[C^\infty(\Omega)]^d \ni \varphi \mapsto \Psi(x,y) \in L^2(\Omega; W)$, such that

$$\mathrm{div}_x\left(\varphi(x)\right) = \mathrm{div}_x\left(\int_Q (a^{(1)})^{1/2}(y)\Psi(x,y)\; \mathrm{d}y\right) \tag{3.2.6}$$

and there exists a constant C such that for fixed $u \in U$

$$\left|\int_\Omega\int_Q Tw \cdot \Psi \; \mathrm{d}y\mathrm{d}x\right| \leq C\left(\int_\Omega |\varphi(x)|^2 \; \mathrm{d}x\right)^{1/2}. \tag{3.2.7}$$

Indeed, if this is true then for fixed $\varphi(x) \in [C^\infty(\Omega)]^d$ there exists $\Psi(x,y) \in$

$L^2(\Omega; W)$ such that (3.2.6) holds. Therefore, choosing Ψ as the test function in (3.2.5) we arrive at

$$\int_\Omega \int_Q Tw(x,y) \cdot \Psi(x,y)\, dx\, dy = -\int_\Omega w(x) \mathrm{div}_x(\varphi(x))\ \mathrm{d}x. \tag{3.2.8}$$

Now the linear functional $\ell : C^\infty(\Omega) \to \mathbb{R}$, $\varphi \mapsto \ell(\varphi)$ defined by

$$\ell(\varphi) := \int_\Omega \int_Q Tw(x,y) \cdot \Psi(x,y)\, dx\, dy,$$

is bounded, by (3.2.7). Therefore, by the Riesz representation theorem, there exists a $v \in L^2(\Omega)$ such that

$$\ell(\varphi) = \int_\Omega v(x)\varphi(x)\ \mathrm{d}x, \quad \forall \varphi \in C^\infty(\Omega).$$

This implies, by the definition of ℓ,

$$\int_\Omega v(x)\varphi(x)\ \mathrm{d}x = \int_\Omega \int_Q Tw(x,y) \cdot \Psi(x,y)\, dx\, dy. \tag{3.2.9}$$

Equations (3.2.8) and (3.2.9) imply

$$\int_\Omega v(x)\varphi(x)\ \mathrm{d}x = -\int_\Omega w(x)\mathrm{div}_x(\varphi(x))\ \mathrm{d}x, \quad \forall \varphi \in C^\infty(\Omega).$$

Therefore $w \in H_0^1(\Omega)$ with $\nabla_x w = v$. Hence $w \equiv 0$ since $w \in H_0^1 \cap (H_0^1)^\perp = \{0\}$.

It remains to prove (3.2.6) and (3.2.7). For fixed $\varphi(x) \in C^\infty(\Omega)$ let $\varphi_0(x) \in H_0^1(\Omega)$ be the unique solution to

$$-\mathrm{div}_x \left(a^{\mathrm{hom}} \nabla_x \varphi_0 - \varphi\right) = 0, \tag{3.2.10}$$

where a^{hom} is given by Lemma 2.1.9. It is clear that φ_0 exists, is unique and $\varphi_0 \in C^\infty(\Omega)$. Next, take $\varphi_1 \in C^\infty(\Omega; H^1_\#)$ to be a solution to

$$-\mathrm{div}_y\Big(a^{(1)}(y)\left[\nabla_y \varphi_1(x,y) + \nabla_x \varphi_0(x)\right]\Big) = 0,$$

existence is guaranteed by the key assumption (3.2.1).

Setting

$$\Psi(x,y) := (a^{(1)})^{1/2}(y)\left[\nabla_y\varphi_1(x,y)+\nabla_x\varphi_0(x)\right], \tag{3.2.11}$$

we see, by construction, $\Psi \in C^\infty(\Omega; W)$ and

$$\int_Q (a^{(1)})^{1/2}(y)\Psi(x,y)\ \mathrm{d}y = \int_Q a^{(1)}(y)\left[\nabla_y\varphi_1(x,y)+\nabla_x\varphi_0(x)\right]\ \mathrm{d}y = a^{\mathrm{hom}}\nabla_x\varphi_0(x), \tag{3.2.12}$$

where the last equality comes from noticing that $\varphi_1(x,y) = N^r(y)\nabla_x\varphi_0(x)$ for $N^r(y)$ the solution to classical cell problem (3.2.3). Equations (3.2.10) and (3.2.12) imply (3.2.6).

Inequality (3.2.7) results from the following observations: for fixed $\eta \in L^2(\Omega; W)$

$$\begin{aligned}\left|\int_\Omega\int_Q \eta\cdot\Psi\ \mathrm{d}y\mathrm{d}x\right| &= \int_\Omega\int_Q \eta\cdot(a^{(1)})^{1/2}(y)\nabla_x\varphi_0(x)\ \mathrm{d}y\mathrm{d}x\\ &\le \left(\int_\Omega|\nabla_x\varphi_0|^2\right)^{1/2}\left(\int_\Omega\int_Q|(a^{(1)})^{1/2}(y)\eta|^2\ \mathrm{d}y\mathrm{d}x\right)^{1/2},\end{aligned}$$

and, by (3.2.10) and the positivity of a^{hom}, there exists a constant C such that

$$\begin{aligned}\int_\Omega|\nabla_x\varphi_0|^2 \le C\int_\Omega a^{\mathrm{hom}}\nabla_x\varphi_0\cdot\nabla_x\varphi_0 &= C\int_\Omega \varphi\cdot\nabla_x\varphi_0\\ &\le C\left(\int_\Omega|\nabla_x\varphi_0|^2\right)^{1/2}\left(\int_\Omega|\varphi|^2\right)^{1/2}.\end{aligned}$$

3.2.2 High contrast regime.

For $n=1$, $d\ge 2$ and isolated inclusions Q_0 consider in (3.1.2) both $a^{(1)}(y)$ and $a^{(0)}(y)$ to be uniformly positive and supported in Q_1 and Q_0 respectively, i.e. $a^{(1)}(y) = \chi_1(y)a^{(1)}(y)$ and $a^{(0)}(y) = \chi_0(y)a^{(0)}(y)$. This is the high contrast regime reviewed in Section 2.2 and we shall prove the high contrast homogenisation theorem stated therein using the new tools outlined in Section 3.1. To this end, the space of microscopic oscillations V, see (3.1.11), by the connectedness of Q_1, is of the form

$$V := \left\{v\in H^1_\#(Q) : v(y) = c+\chi_0(y)w(y), \text{for some constant } c \text{ and } w\in H^1_0(Q_0)\right\}. \tag{3.2.13}$$

Under the following equivalent norm on $H^1_\#(Q)$

$$\|u\|_H^2 := \left(\int_{Q_1} u\right)^2 + \int_Q |\nabla_y u|^2,$$

the key assumption (3.1.20) holds if the following inequality holds: There exists a constant $C > 0$ such that for all $u \in H^1_\#(Q)$ there exists $v \in V$ such that

$$\|u - v\|_H^2 \leq C \int_{Q_1} |\nabla_y u|^2 \, \mathrm{d}y. \tag{3.2.14}$$

Let us now show this to be the case. For fixed $u \in H^1_\#(Q)$, denote $\tilde{u}$ to be the harmonic extension of u, see Lemma C.0.5. Defining $v := u - \tilde{u} + \frac{1}{|Q_1|}\int_{Q_1} \tilde{u} \, \mathrm{d}y$ we see that $v \in V$ and

$$\|u - v\|_H^2 = \|\tilde{u} - \tfrac{1}{|Q_1|}\int_{Q_1} \tilde{u} \, \mathrm{d}y\|_H^2 = \int_Q |\nabla_y \tilde{u}|^2 \leq c \int_{Q_1} |\nabla_y u|^2,$$

where the last inequality comes from the properties of the harmonic extension, see Appendix C, Lemma C.0.5. Hence (3.2.14) holds.

As the key assumption holds, we conclude the two-scale homogenisation Theorem 3.1.7 holds. To show this coincides with the homogenisation theorem in Section 2.2, Theorem 2.2.4 we have to show that $u^0 \in U$ if, and only if, $u^0(x, y) = u(x) + v(x, y)$ for some $u \in H^1_0(\Omega)$ and $v \in L^2(\Omega; H^1_0(Q_0))$. Furthermore, we must show

$$\int_\Omega \int_Q Tu \cdot Tw \, \mathrm{d}y\mathrm{d}x = \int_\Omega a^{\mathrm{hom}} \nabla_x u \cdot \nabla_x w \, \mathrm{d}x \qquad \forall u, w \in H^1_0(\Omega), \tag{3.2.15}$$

where $a^{\mathrm{hom}}(y)$ is the homogenised matrix for a perforated domain given by Lemma 2.2.1.

For fixed $u \in H^1_0(\Omega)$, $v \in L^2(\Omega; H^1_0(Q_0))$, let $u^0(x, y) := u(x) + v(x, y)$. As in Section 3.2.1, we find $u \in U$ for

$$Tu = \left(a^{(1)}(y)\right)^{1/2} \left[\nabla_x u + \nabla_y u_1\right],$$

where u_1 solves the problem

$$-\mathrm{div}_y\Big(a^{(1)}(y)\,[\nabla_y u_1 + \nabla_x u]\Big) = 0, \quad \text{in } Q.$$

Since $v \in L^2(\Omega; H_0^1(Q_0))$, we observe that

$$\int_\Omega \int_Q v(x,y)\cdot \mathrm{div}_x\Big(\,\big(a^{(1)}(y)\big)^{1/2}\,\Psi(x,y)\Big)\;\mathrm{d}y\mathrm{d}x = 0, \quad \forall \Psi \in C^\infty(\Omega; W).$$

Therefore $v \in U$ and $Tv = 0$. This implies $u^0 \in U$ and $Tu^0 = Tu$.

Let us now prove the converse inclusion. Fix $u^0 \in U$. By (3.2.13), we can directly see that $u^0(x,y) = u(x) + v(x,y)$ for some $u \in L^2(\Omega)$, $v \in L^2(\Omega; H_0^1(Q_0))$. It remains to show $u \in H_0^1(\Omega)$. Using the approach in Section 3.2.1 it is sufficient to prove, for fixed $\varphi \in C^\infty(\Omega)$ we can construct a $\Psi \in C^\infty(\Omega; W)$, $\Psi = 0$ in Q_0 and

$$\mathrm{div}_x\,(\varphi(x)) = \mathrm{div}_x\left(\int_{Q_1} (a)^{1/2}(y)\Psi(x,y)\;\mathrm{d}y\right)$$

and there exists a constant C such that for fixed $u \in U$

$$\left|\int_\Omega \int_Q Tw\cdot\Psi\;\mathrm{d}y\mathrm{d}x\right| \le C\left(\int_\Omega |\varphi(x)|^2\;\mathrm{d}x\right)^{1/2}.$$

The important point is $\Psi = 0$ in Q_0 and therefore

$$\int_\Omega \int_Q u^0(x,y)\mathrm{div}_x\Big(\,\big(a^{(1)}(y)\big)^{1/2}\,\Psi(x,y)\Big)\;\mathrm{d}y\mathrm{d}x$$
$$= \int_\Omega \int_{Q_1} u(x)\mathrm{div}_x\Big(a^{(1)}(y)\Psi(x,y)\Big)\;\mathrm{d}y\mathrm{d}x$$

We guarantee such a Ψ by repeating the arguments in Section 3.2.1 for $a^{(1)}(y) = \chi_1(y)a(y)$, with $a > \nu I$ and replacing the classical homogenised matrix by the perforated domain's homogenised limit a^{hom}, given in Section 2.2. It is also clear that $Tu^0 = Tu$ and (3.2.15) holds.

Chapter 4

Partially degenerating Elastic inclusions

In this chapter we shall study our first physical example of a partially degenerating PDE system that arises from studying the deformations for a particular linear elastic composite material. The composite in question shall be a two phase material whose 'inclusion' phase is disjoint and periodically distributed through the 'matrix' phase. While the matrix phase is an arbitrary anisotropic material with a uniformly positive elasticity tensor, the inclusion phase is considered to be isotropic and 'soft' in shear; namely the shear modulus for the inclusion material is chosen to be of the order ε^2, where ε is the composite's periodicity size, while the bulk modulus remains uniformly positive. The elasticity equations for this composite are hence a system of partially degenerating PDEs as $\varepsilon \to 0$. Panasenko studied via the method of asymptotic expansions, in [28, 30], the elasticity equations, for such elastic composites, with externally applied macroscopic body forces.

The purpose of our study is to find and analyse the two-scale homogenised limit for the above mentioned elasticity equations with, possibly microscopic body forces, i.e. body forces that depend on the fast variable $y = x/\varepsilon$. Furthermore, we shall study the spectrum of the homogenised limit operator and some related spectral questions. We find that the homogenised limit elastostatic equations have the novel feature of their structure being of a classical or high contrast limit form depending upon the microscopic nature of the external body force. Such properties have interesting consequences, especially when looking at the limit spectrum. In Chapter 2 the introduction of high contrasts gave rise to microresonances which, in a sense, are responsible for the structure of the limit spectrum including the presence of gaps in the case $\Omega = \mathbb{R}^n$. We find for this particular partially high contrast problem that while the homogenised limit is of a genuine two-scale nature there are no gaps. The problem formulation and the main results mentioned above are formulated in Section 4.1. Section 4.2 is dedicated to the study of the space of microscopic oscillations V and to the proof of the key assumption (3.1.20) sufficient to allow passing to the homogenised limit. In Section 4.3 we pass to the homogenised limit and explain the external body force dependence of the said limit. In Section 4.4 we study the spectrum of the two-scale limit operator and prove that the original spectrum converges, in the sense of Hausdorff, to the limit spectrum. For this we additionally prove a key two-scale compactness property for the present example.

4.1 Problem formulation and main results

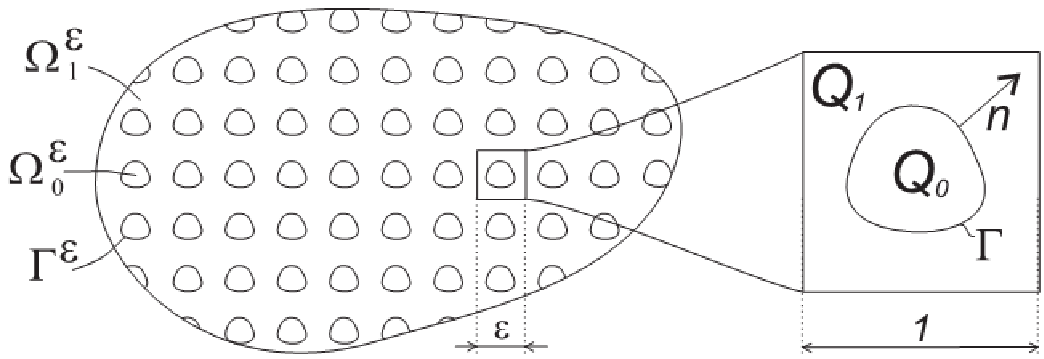

Figure 4.1:

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$,open and bounded, be the domain occupied by the composite material, see figure 4.1. Let $Q = [0,1)^d$ be the periodic reference cell. Q consists of two disjoint regions: the 'inclusion' Q_0, a bounded subset of Q with smooth boundary Γ and the 'matrix' $Q_1 = Q \backslash \overline{Q_0}$. We assume that Q_1 is connected and assume Q_0 is strictly contained in Q; i.e. there exists a compact

K such that $Q_0 \subset K \subset Q$. We denote by F_0 the Q-periodic extension of Q_0 throughout $\mathbb{R}^d$, i.e.

$$F_0 := \left\{ x : \; x = y + k \text{ for some } y \in Q_0 \text{ and some } k \in \mathbb{Z}^d \right\},$$

and denote by εF_0 the ε contraction of F_0, i.e., $\varepsilon F_0 = \{x : x/\varepsilon \in F_0\}$. We denote by Ω_0^ε and Ω_1^ε the inclusion phase and matrix phase respectively. That is, $\Omega_0^\varepsilon = \Omega \cap \varepsilon F_0$ and $\Omega_1^\varepsilon = \Omega \backslash \Omega_0^\varepsilon$.

We shall consider first the following 'resolvent' problem:

$$\begin{aligned} -\text{div}\,(C^\varepsilon e(u)) + \alpha u = f^\varepsilon \quad & \text{in } \Omega \\ u = 0 \quad & \text{on } \partial\Omega. \end{aligned} \tag{4.1.1}$$

$\alpha \geq 0$. The underlying density function $\rho(y)$ is assumed identically equal to unity, for simplicity. Here $u \in [H_0^1(\Omega)]^d$ is the unknown displacement, $e(u)_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is the infinitesimal strain tensor . Furthermore $C^\varepsilon(x) = C(x/\varepsilon)$ for $C(y) \in [L^\infty_\#(Q)]^{d\times d\times d\times d}$, is the elasticity tensor of the composite material which is considered to be an arbitrary positive definite elasticity tensor in the matrix, while to be an isotropic tensor in the inclusion with Lamé coefficients $\lambda \sim O(1)$, $\mu \sim O(\varepsilon^2)$. Explicitly $C(y)$ is of the form:

$$\begin{gathered} C(y) = C^{(1)}(y) + \varepsilon^2 C^{(0)}(y), \\ C^{(1)}_{ijpq}(y) = \chi_1(y) C^{(2)}_{ijpq}(y) + \chi_0(y)\delta_{ij}\delta_{pq},\; C^{(0)}_{ijpq}(y) = \chi_0(y)(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}). \end{gathered} \tag{4.1.2}$$

(We have set, for simplicity, $\lambda = 1$, $\mu = \varepsilon^2$ in Q_0). Here δ is the Kronecker delta symbol, χ_i is the characteristic function of Q_i, $C^{(2)} \in [L^\infty_\#(Q)]^{d\times d\times d\times d}$ is taken to be symmetric and positive definite:

$$C^{(2)}_{ijpq}(y) = C^{(2)}_{jipq}(y) = C^{(2)}_{pqij}(y), \qquad C^{(2)}_{ijpq}(y)\eta_{pq}\eta_{ij} \geq \nu|\eta|^2, \tag{4.1.3}$$

for some $\nu > 0$, for all $y \in Q_1$, for all symmetric η. $f^\varepsilon(x) \in [L^2(\Omega)]^d$ is a prescribed externally applied 'body force'.

We can see, from (4.1.2), $C^{(1)}$ is symmetric and non-negative while $C^{(1)} + C^{(0)}$

is symmetric and positive definite, i.e.

$$C^{(i)}_{ijpq}(y) = C^{(i)}_{jipq}(y) = C^{(i)}_{pqij}(y), \quad \text{for } i = 0, 1. \tag{4.1.4}$$
$$C^{(1)}_{ijpq}(y)\eta_{pq}\eta_{ij} \geq 0, \quad \left(C^{(1)}_{ijpq}(y) + C^{(0)}_{ijpq}(y)\right)\eta_{pq}\eta_{ij} \geq \nu|\eta|^2,$$

for some $\nu > 0$, for all $y \in Q$, for all symmetric η. That is, $C(y)$ is a tensor of a partially degenerate type, see Chapter 3.

The weak formulation for problem (4.1.1) is: Find $u^\varepsilon \in [H^1_0(\Omega)]^d$ such that

$$\int_\Omega C^{(1)}(\tfrac{x}{\varepsilon})e(u^\varepsilon)\cdot e(\phi) + \varepsilon^2 \int_{\Omega_0^\varepsilon} C^{(0)}(\tfrac{x}{\varepsilon})e(u^\varepsilon)\cdot e(\phi) + \alpha \int_\Omega u^\varepsilon \cdot \phi = \int_\Omega f^\varepsilon \cdot \phi,$$
$$\forall \phi \in [H^1_0(\Omega)]^d. \tag{4.1.5}$$

The corresponding quadratic form

$$\mathcal{A}^\varepsilon(u,v) := \int_\Omega C^{(1)}(\tfrac{x}{\varepsilon})e(u)\cdot e(v) + \varepsilon^2 \int_{\Omega_0^\varepsilon} C^{(0)}(\tfrac{x}{\varepsilon})e(u)\cdot e(v) + \alpha \int_\Omega u \cdot v$$

is positive, which is ensured by (strong) ellipticity of $C^{(1)}+C^{(0)}$. This tells us that the spectrum of the self-adjoint operator A^ε, defined by the quadratic form $\mathcal{A}^\varepsilon$, is a subset of the positive real line. For fixed $\alpha \geq 0$, fixed $\varepsilon > 0$, the existence and uniqueness of a solution to problem (4.1.1) is guaranteed by the Korn's inequality and Lax-Milgram Lemma. We shall denote this solution by u^ε.

We shall now study the behaviour of u^ε for any sequence ε tending to zero. The two-scale limit u^0 of u^ε and the structure of its corresponding limit problem are found to be highly dependent on the microscopic behaviour of the body force f^ε. Namely, if the external body force is microscopically irrotational we have the following homogenisation result.

Theorem 4.1.1 (**First Main Homogenisation Result**). *Let $f^\varepsilon(x)$ weakly two-scale converge to $f(x,y) = f_0(x) + \nabla_y f_1(x,y)$ for given, sufficiently regular, f_0, f_1. Then the sequence u^ε, of solutions to* (4.1.5)*, strongly converges to $u(x)$ in $L^2(\Omega)$ as $\varepsilon \to 0$, where $u \in [H^1_0]^d$ is the unique solution to*

$$-\text{div}\left(C^{\text{hom}}\nabla u(x)\right) + \alpha u(x) = < f > (x) \quad \textit{in } \Omega. \tag{4.1.6}$$

C^{hom} *is the constant coefficient positive homogenised tensor given by*

$$C^{\text{hom}}_{ijrs} = \int_Q C^{(1)}_{ijpq}(y) \left(\delta_{pr}\delta_{qs} + \frac{\partial N^p_{rs}}{\partial y_q} \right) \mathrm{d}y. \tag{4.1.7}$$

Here $N_{rs} = (N^1_{rs}, N^2_{rs}, \ldots, N^d_{rs})$ *is a* Q*-periodic solution to the degenerate cell problem*

$$-\mathrm{div}_y \left(C^{(1)}(y) \left(e_r \otimes e_s + \nabla_y N_{rs}(y) \right) \right) = 0 \quad \text{in } Q. \tag{4.1.8}$$

Remark. The solutions N_{rs} to the degenerate cell problem (4.1.8) are not unique, but the homogenised matrix C^{hom} as defined by (4.1.7) is unique.

Theorem 4.1.1 says in the case of (partially) high contrasting coefficients we can arrive at a classical homogenised limit problem; the limit problem (4.1.6) is a one-scale problem, independent of the microscopic variable, i.e. the limit solution u^0 has no microscopic oscillations. Furthermore, the strong convergence, in L^2, of u^ε to u is guaranteed even though $f^\varepsilon(x) \overset{2}{\rightharpoonup} f(x,y)$ weakly. These striking results are very different to the case of 'full' highly contrasting coefficients where we have seen, in Section 2.2, that the homogenised limit is of a genuine two-scale nature, for a general body force f^ε, with the limit function depending on both the macroscopic and microscopic variable. The reason for the limit function having no microscopic variables, for the described body force, is precisely due to the form of the partial degeneracy. As mentioned above the inclusion phase was chosen to be isotropic with Lamé coefficient $\lambda \sim 0(1)$, which means that in the asymptotic limit $\varepsilon \to 0$ the inclusion phase is microscopically incompressible, i.e. the degeneracy imposes the constraint $\mathrm{div}_y u^0 = 0$, and we see that, as a consequence, if we choose microscopically compressing body forces no deformations will occur on the microscopic scale. For all other body forces, in particular microscopically rotational ones, microscopic deformations do occur and the homogenised limit problem is of a genuine two-scale nature. This is the second formulation of the homogenisation result.

Theorem 4.1.2 (Second Main Homogenisation Result). *Let* $f^\varepsilon(x)$ *weakly (strongly) two-scale converge to* $f(x,y)$ *as* $\varepsilon \to 0$*. Then the sequence* u^ε *weakly (strongly) two-scale converges to* $u^0(x,y) = u(x) + v(x,y)$ *as* $\varepsilon \to 0$*, where* $(u,v) \in$

$[H_0^1(\Omega)]^d \times [L^2(\Omega; H_0^1(Q_0))]^d$ *is the unique solution to*

$$-\text{div}\left(C^{\text{hom}}\nabla u(x)\right) + \alpha u(x) + \alpha < v > (x) =< f > (x) \quad \textit{in } \Omega, \tag{4.1.9}$$

$$\begin{aligned} -\Delta_y v(x,y) + \alpha v(x,y) &= f(x,y) + \nabla_y p(x,y) \quad \textit{in } Q_0 \\ \text{div}_y v(x,y) &= 0 \quad \textit{in } Q_0 \\ v(x,y) &= 0 \quad \textit{on } \partial Q_0, \end{aligned} \tag{4.1.10}$$

where $p \in L^2(\Omega; H^1(Q_0))$ *is unknown.* C^{hom} *is the constant coefficient positive tensor given by*

$$C^{\text{hom}}_{ijrs} = \int_Q C^{(1)}_{ijpq}(y)\left(\delta_{pr}\delta_{qs} + \frac{\partial N^p_{rs}}{\partial y_q}\right) \mathrm{d}y.$$

Here $N_{rs} = (N^1_{rs}, N^2_{rs}, \dots, N^d_{rs})$ *is a* Q*-periodic solution to the degenerate cell problem*

$$-\text{div}_y\left(C^{(1)}(y)\left(e_r \otimes e_s + \nabla_y N_{rs}(y)\right)\right) = 0 \quad \textit{in } Q.$$

Hence we find, by Theorem 4.1.2, that the homogenised limit solution depends on the microscopic variable y if the solution $v(x,y)$ to problem (4.1.10) is not trivial. If, for example, we have $f^\varepsilon(x) = f(x, x/\varepsilon)$ for a non-zero $f(x,y)$, not irrotational in y, problem (4.1.10) is the inhomogeneous Stokes problem and is well known to have a non-trivial solution, see e.g. [36].

Denoting A^0, defined at the end Section 3.1, to be the homogenised limit operator given by Theorem 4.1.2, the homogenisation theorem states that A^ε converges to A^0 as $\varepsilon \to 0$ in the sense of strong two-scale resolvent convergence. A consequence of strong resolvent two-scale convergence, see Section B.1 in Appendix B, is that if λ^0 is in the spectrum of A^0, denoted $\sigma(A^0)$, there exists λ^ε belonging to the spectrum of A^ε, denoted $\sigma(A^\varepsilon)$, such that $\lambda^\varepsilon \to \lambda^0$ as $\varepsilon \to 0$. We wish to study the asymptotic behaviour of the eigenvalues of the spectrum $\sigma(A^\varepsilon)$, i.e. we wish to show that if λ^ε is an eigenvalue of A^ε and $\lambda^\varepsilon \to \lambda$ as $\varepsilon \to 0$ then λ belongs to the spectrum $\sigma(A^0)$. This spectral compactness result is our other main result. (For this to hold, the inclusions in Ω_0^ε intersecting $\partial\Omega$ have to be 'removed', e.g. to be replaced by a matrix material.)

Lemma 4.1.3 (Spectral compactness Lemma)**.** *The spectrum of* A^ε, $\sigma(A^\varepsilon)$, *converges in the sense of Hausdorff to the spectrum of* A^0, $\sigma(A^0)$. *That is*

1. *For every $\lambda \in \sigma(A^0)$ there exists $\lambda_\varepsilon \in \sigma(A^\varepsilon)$ such that $\lambda_\varepsilon \to \lambda$.*

2. *If there exists $\lambda_\varepsilon \in \sigma(A^\varepsilon)$ such that $\lambda_\varepsilon \to \lambda$, then $\lambda \in \sigma(A^0)$.*

This result tells us that if we wish to study the limit behaviour of the eigenvalues of A^ε as $\varepsilon \to 0$ then it is sufficient to study the spectrum $\sigma(A^0)$. To this end, let (λ, u^0) be an eigenvalue-eigenfunction pair of A^0. Then $u^0(x,y) = u(x) + v(x,y)$ satisfies

$$-\text{div}\left(C^{\text{hom}}\nabla u(x)\right) = \lambda u(x) + \lambda < v > (x) \quad \text{in } \Omega, \tag{4.1.11}$$

$$\begin{aligned} -\Delta_y v(x,y) &= \lambda v(x,y) + \nabla_y p(x,y) \quad \text{in } Q_0 \\ \text{div}_y v(x,y) &= 0 \quad \text{in } Q_0 \\ v(x,y) &= 0 \quad \text{on } \partial Q_0. \end{aligned} \tag{4.1.12}$$

Notice here that the equations (4.1.11)-(4.1.12) are not coupled: this uncoupled nature is due to the fact that $\lambda u(x)$ originally present on the right hand side of (4.1.12) can be absorbed by $p(x,y)$ when Q_0 is disjoint, i.e. $u(x) = \nabla_y q(x,y)$ for some $q(x,y)$. The main consequence of this uncoupling is that the spectrum of A^0 is simply the union of the spectra for the operators defined by (4.1.11) and (4.1.12), i.e.

Lemma 4.1.4 (Spectrum of the limit operator)**.** *The spectrum of the homogenised limit operator A^0, $\sigma(A^0)$ has the following representation:*

$$\sigma(A^0) = \{\lambda_n \mid n \in \mathbb{N}\} \cup \{\mu_n \mid n \in \mathbb{N}\},$$

where λ_n satisfy, for some non-trivial $u_n \in H_0^1(\Omega)$,

$$-\text{div}\left(C^{\text{hom}}\nabla u_n(x)\right) = \lambda_n u_n(x) \quad \text{in } \Omega,$$

and μ_n satisfy, for some non-trivial $v_n \in H_0^1(Q_0)$, $p_n \in H^1(Q_0)$,

$$\begin{aligned} -\Delta_y v_n(y) &= \mu_n v_n(y) + \nabla_y p_n(y) \qquad && \text{in } Q_0\ , \\ \text{div}_y v_n(y) &= 0 && \text{in } Q_0. \end{aligned}$$

Remark. In the case of $\Omega = \mathbb{R}^d$ the operator $L := -\text{div}\left(C^{\text{hom}}\nabla u_n(x)\right)$ has essential spectrum coinciding with the positive real line. The spectrum of A^0 is still the union of the spectra for the operators defined by (4.1.11) and (4.1.12), which implies that the spectrum of A^0 contains no spectral gaps and has embedded eigenvalues μ_n corresponding to the eigenvalues of the Stokes problem on the inclusion Q_0. Furthermore, the spectral compactness result, Lemma 4.1.3 holds in this case, cf. [37, 38].

4.2 Space of Microscopic Oscillations

The space V of microscopic oscillations, cf. (3.1.11) is

$$V := \left\{ v \in [H^1_\#(Q)]^d : C^{(1)}(y)\nabla_y v(y) = 0 \right\}.$$

By the structure of $C^{(1)}$, see (4.1.2), it can be directly seen that $v \in V$ if, and only if, v is constant in Q_1 and $\mathrm{div}_y v = 0$ in Q_0, i.e.

$$V = \{ v \in [H^1_\#(Q)]^d : v(y) = k + \chi_0(y)w(y) \text{ for some } k \in \mathbb{R}^d,\ w \in [H^1_0(Q_0)]^d, \\ \mathrm{div}_y w = 0\}. \quad (4.2.1)$$

Here, and throughout the chapter, we shall drop the χ_0 and imply w to be extended by zero into Q_1. As outlined in the theory of partially degenerating homogenisation in Chapter 3 it is sufficient to check, for the tensor $C^{(1)}$, that the following key assumption holds, cf. (3.1.20): There exists a constant $c > 0$ such that for any $u \in [H^1_\#(Q)]^d$

$$\|P_{V^\perp} u\|^2_{H^1(Q)} \leq c \left(\|e(u)\|^2_{L^2(Q_1)} + \|\nabla \cdot u\|^2_{L^2(Q_0)} \right). \quad (4.2.2)$$

Here $P_{V^\perp}$ is the orthogonal projection in H^1 on to $V^\perp$, the orthogonal complement of V. It is sufficient to prove the key assumption (4.2.2) for any equivalent H^1 norm. Therefore we shall prove the result for the following equivalent norm

$$\|u\|^2_H := \left(\int_{Q_1} u \, \mathrm{d}x \right)^2 + \int_Q |\nabla u|^2 \, \mathrm{d}x, \quad (4.2.3)$$

which is induced by the following inner product

$$(u, v)_H := \left(\int_{Q_1} u \ \mathrm{d}x\right) \cdot \left(\int_{Q_1} v \ \mathrm{d}x\right) + \int_{Q} \nabla u \cdot \nabla v \ \mathrm{d}x. \qquad (4.2.4)$$

We shall now consider $V^{\perp}$ to be the orthogonal complement of V with respect to the inner product (4.2.4), i.e. $u \in V^{\perp}$ if

$$\left(\int_{Q_1} u \ \mathrm{d}x\right)\left(\int_{Q_1} v \ \mathrm{d}x\right) + \int_{Q} \nabla u \cdot \nabla v \ \mathrm{d}x = 0 \quad \forall v \in V. \qquad (4.2.5)$$

Since constant vectors are in V, then by (4.2.5) $\int_{Q_1} u \ \mathrm{d}x = 0$. Furthermore, the functions $\phi \in [C_0^{\infty}(Q_0)]^d$ with $\mathrm{div}\phi = 0$ belong to V, and by (4.2.5)

$$0 = (u, \phi)_H = \int_{Q} \nabla u \cdot \nabla \phi \ \mathrm{d}x = - < \Delta u, \phi >, \qquad (4.2.6)$$

where $< \Delta u, \phi >$ is that action of the distribution Δu on the test function ϕ. Equation (4.2.6) states the distribution Δu is orthogonal to all divergent free test functions in Q_0. It is well known, see e.g. [36], that such distributions are potentials, i.e. $\Delta u = \nabla \psi$ for some distribution ψ and, since $u \in [H^1(Q_0)]^d$, $\psi \in L^2(Q_0)$. Therefore, we see that $u \in V^{\perp}$ if, and only if,

$$\Delta u = \nabla \psi \quad \text{for some } \psi \in L^2(Q_0) \text{ in } Q_0, \qquad \int_{Q_1} u \ \mathrm{d}x = 0.$$

To prove (4.2.2) it is sufficient to prove the following:

Lemma 4.2.1 (Modification Lemma)**.** *There exists a constant $c > 0$, such that for all $u \in [H^1_{\#}(Q)]^d$ there exists $v \in [H^1_{\#}(Q)]^d$, such that*

(i) $u = v$ *in* Q_1,

(ii) $\mathrm{div}u = \mathrm{div}v$ *in* Q_0,

(iii) $\|\nabla v\|^2_{L^2(Q)} \leq c\left(\|e(u)\|^2_{L^2(Q_1)} + \|\nabla \cdot u\|^2_{L^2(Q_0)}\right)$,

(iv) $\Delta v = \nabla \phi$ *in* Q_0 *for some* $\phi \in L^2(Q_0)$.

For if we define $u_1 := u - v + \frac{1}{|Q_1|}\int_{Q_1} u \ \mathrm{d}y$, where v is the modification of u

given by Lemma 4.2.1, then clearly $u_1 \in V$, $\int_{Q_1} (u - u_1)\ \mathrm{d}y = 0$, and

$$\|P_{V^\perp} u\|_H^2 \leq \|u - u_1\|_H^2 = \left| \int_{Q_1} (u - u_1)\ \mathrm{d}x \right|^2 + \int_Q |\nabla u - \nabla u_1|^2\ \mathrm{d}x$$
$$= \int_Q |\nabla v|^2\ \mathrm{d}x \leq c \left(\|e(u)\|_{L^2(Q_1)}^2 + \|\nabla \cdot u\|_{L^2(Q_0)}^2 \right),$$

where the last inequality follows from Lemma 4.2.1(iii). To prove Lemma 4.2.1 we shall need the fact

Lemma 4.2.2. *There exists a constant $c > 0$ such that for all $u \in L^2(Q_0)$*

$$\|u\|_{L^2(Q_0)} \leq c \left(\|\nabla u\|_{H^{-1}(Q_0)} + \left| \int_{Q_0} u\ \mathrm{d}x \right| \right).$$

Proof. Assume the contrary to be true. That is $\forall\ c > 0$ there exists a $u \in L^2(Q_0)$ such that

$$\|u\|_{L^2(Q_0)} \geq c \left(\|\nabla u\|_{H^{-1}(Q_0)} + \left| \int_{Q_0} u\ \mathrm{d}x \right| \right)$$

Now let $\{u_n\}$ be such a sequence in L^2 that $\|u_n\|_{L^2(Q_0)} = 1$ for all $n \in \mathbb{N}$, and

$$1 = \|u_n\|_{L^2(Q_0)} \geq n \left(\|\nabla u_n\|_{H^{-1}(Q_0)} + \left| \int_{Q_0} u_n\ \mathrm{d}x \right| \right). \tag{4.2.7}$$

Since $L^2(Q_0) \hookrightarrow H^{-1}(Q_0)$ is a compact embedding, $\{u_n\}$ has a convergent subsequence in H^{-1} to a limit u_0 say. After passing to the necessary subsequence we have

$$u_n \longrightarrow u_0 \text{ in } H^{-1}.$$

By Lion's Lemma, see C.0.4,

$$\|u_n - u_m\|_{L^2} \leq c \left(\|\nabla u_n\|_{H^{-1}} + \|\nabla u_m\|_{H^{-1}} + \|u_n - u_m\|_{H^{-1}} \right) \quad n, m \in \mathbb{N},$$

which via (4.2.7) implies $\{u_n\}$ is a Cauchy sequence in L^2 and, therefore, by the completeness of L^2, converges to a limit $\bar{u}$ say. Since L^2 is embedded into H^{-1} we have

$$\|u_n - \bar{u}\|_{H^{-1}} \leq c \|u_n - \bar{u}\|_{L^2} \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

which implies $\bar{u} = u_0$, and also $\int_{Q_0} u_n\ \mathrm{d}x \longrightarrow \int_{Q_0} u_0\ \mathrm{d}x$. Furthermore, $\nabla u_n \longrightarrow$

∇u_0 in H^{-1} since: for $i = 1, \cdots, n$,

$$\begin{aligned} | < u_{n,i} - u_{0,i}, v \ >| &= |- < u_n - u_0, v_{,i} >| \\ &\le c\|v_{,i}\|_{L^2}\|u_n - u_0\|_{L^2} \qquad \forall v \in H_0^1(Q_0). \end{aligned}$$

From (4.2.7) we see $\|\nabla u_n\|_{H^{-1}} \longrightarrow 0$, $\left|\int_{Q_0} u_n \ \mathrm{d}x\right| \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore

$$\|\nabla u_0\|_{H^{-1}} = 0, \quad \int_{Q_0} u_0 \ \mathrm{d}x = 0,$$

which implies $u_0 = 0$. Hence a contradiction as, from the construction of $\{u_n\}$, $\|u_0\|_{L^2} = 1$. □

Proof of Lemma 4.2.1. For fixed $u \in [H^1_\#(Q)]^d$, decomposing $u = u_1 + u_2$ for some $u_1 \in V$, $u_2 \in V^\perp$. Define $v := u_2 + k$, where k is the value of u_1 in Q_1. Clearly v satisfies (i) and (ii), as, since $u_1 \in V$,

$$v(y) = u_2(y) + k = u_2(y) + u_1(y) = u(y), \qquad\qquad y \in Q_1$$

and

$$\mathrm{div}_y v = \mathrm{div}_y u_2 = \mathrm{div}_y u_2 + \mathrm{div}_y u_1 = \mathrm{div}_y u \qquad\qquad \text{in } Q_0.$$

It remains to show that there exists a constant $c > 0$ such that

$$\int_Q |\nabla u_2|^2 \ \mathrm{d}x \le c\left(\int_{Q_1} |e(u)|^2 \ \mathrm{d}x + \int_{Q_0} |\nabla \cdot u|^2 \ \mathrm{d}x\right) \quad \forall u \in [H^1_\#(Q)]^d. \quad (4.2.8)$$

For fixed $w \in V^\perp$, let $\tilde{w}$ be its harmonic extension, see Lemma C.0.5. Denote $w_1 := w - \tilde{w}$. Evidently, $w = \tilde{w} + w_1$, $w_1 \in [H^1_0(Q_0)]^d$, $\int_{Q_1} w \ \mathrm{d}x = 0$, and

$$\begin{aligned} \tilde{w} &= w \text{ in } Q_1, \qquad & \Delta \tilde{w} &= 0 \text{ in } Q_0, \\ w_1 &= 0 \text{ in } Q_1, \qquad & \Delta w_1 &= \nabla \varphi \text{ in } Q_0, \end{aligned}$$

where $\varphi \in L^2(Q_0)$. Since

$$\int_Q |\nabla w|^2 \ \mathrm{d}x \le \int_Q |\nabla \tilde{w}|^2 \ \mathrm{d}x + \int_{Q_0} |\nabla w_1|^2 \ \mathrm{d}x,$$

and

$$\int_{Q_0} |\nabla \cdot w_1|^2 \leq \int_{Q_0} |\nabla \cdot w|^2 + \int_{Q} |\nabla \tilde{w}|^2,$$

to show inequality (4.2.8), it is sufficient to prove the following inequalities

$$\int_{Q} |\nabla \tilde{w}|^2 \ \mathrm{d}x \leq c \left(\int_{Q_1} |e(w)|^2 \ \mathrm{d}x \right), \tag{4.2.9}$$

$$\int_{Q_0} |\nabla w_1|^2 \ \mathrm{d}x \leq c \left(\int_{Q_0} |\nabla \cdot w_1|^2 \ \mathrm{d}x \right). \tag{4.2.10}$$

Inequality (4.2.9) directly follows from Lemma C.0.5.(iii) and the Korn's inequality, see Lemma C.0.7.

Let us now show inequality (4.2.10) holds: let $w_n \in C_0^\infty(Q_0)$ such that $w_n \to w_1$ strongly in H^1. Then, by integration by parts and Lemma 4.2.2

$$\begin{aligned}
\int_{Q_0} \nabla w_n \nabla w_1 \ \mathrm{d}x &= - < w_n, \Delta w_1 >= - < w_n, \nabla \varphi > \\
&= \int_{Q_0} \varphi \nabla \cdot w_n \ \mathrm{d}x \\
&\leq \|\varphi\|_{L^2(Q_0)} \|\nabla \cdot w_n\|_{L^2(Q_0)} \\
&\leq c \|\nabla \cdot w_n\|_{L^2(Q_0)} \left[\|\nabla \varphi\|_{H^{-1}(Q_0)} + \left| \int_{Q_0} \varphi \ \mathrm{d}x \right| \right].
\end{aligned} \tag{4.2.11}$$

For fixed $w \in H_0^1(Q_0)$

$$\begin{aligned}
\|\Delta w\|_{H^{-1}} &= \sup_{\substack{u \in H^1(Q_0) \\ \|u\|_{H^1(Q_0)}=1}} \left| \int_{Q_0} \nabla w \cdot \nabla u \right| \\
&\leq \sup_{\substack{u \in H^1(Q_0) \\ \|u\|_{H^1(Q_0)}=1}} \|\nabla w\|_{L^2(Q_0)} \|\nabla u\|_{L^2(Q_0)} \\
&\leq \sup_{\substack{u \in H^1(Q_0) \\ \|u\|_{H^1(Q_0)}=1}} \|\nabla w\|_{L^2(Q_0)} \|u\|_{H^1(Q_0)} \\
&= \|\nabla w\|_{L^2(Q_0)} = \|w\|_{H_0^1(Q_0)},
\end{aligned}$$

which implies Δ defines a bounded linear operator from H_0^1 to H^{-1}. Therefore

$$\|\nabla\varphi\|_{H^{-1}(Q_0)} = \|\Delta w_1\|_{H^{-1}(Q_0)} \leq \|\nabla w_1\|_{L^2(Q_0)}.$$

Without loss of generality, because adding a constant to φ does not affect $\nabla\varphi$, we can choose $\int_{Q_0}\varphi\,\mathrm{d}x = 0$. Hence (4.2.11) becomes

$$\int_{Q_0}\nabla w_n\nabla w_1\,\mathrm{d}x \leq c\left(\int_{Q_0}|\nabla\cdot w_n|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{Q_0}|\nabla w_1|^2\,\mathrm{d}x\right)^{\frac{1}{2}},$$

Passing to the limit $n\to\infty$ gives the desired result:

$$\int_{Q_0}|\nabla w_1|^2\,\mathrm{d}x \leq c\left(\int_{Q_0}|\nabla\cdot w_1|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{Q_0}|\nabla w_1|^2\,\mathrm{d}x\right)^{\frac{1}{2}},$$

implying (4.2.10). □

4.3 Homogenisation

The weak form of (4.1.1) is stated as follows: Find $u^\varepsilon\in[H_0^1(\Omega)]^d$ such that

$$\int_\Omega C^{(1)}(\tfrac{x}{\varepsilon})e(u^\varepsilon)\cdot e(\phi)+\varepsilon^2C^{(0)}(\tfrac{x}{\varepsilon})e(u^\varepsilon)\cdot e(\phi)+\alpha u^\varepsilon\cdot\phi=\int_\Omega f^\varepsilon\cdot\phi \quad \forall\phi\in[H_0^1(\Omega)]^d. \tag{4.3.1}$$

The quadratic form defined by the left hand side of (4.3.1) is positive definite, since $C^{(1)}+C^{(0)}$ is positive definite, which, along with the Korn's inequality in Ω, see Lemma C.0.6, ensures the existence and uniqueness of $u^\varepsilon\in[H_0^1(\Omega)]^d$ for fixed $\alpha\geq 0$. We wish to see how u^ε behaves as ε tends to zero. Substituting $\phi=u^\varepsilon$ into (4.3.1) and using (4.1.4) we see that there exists a constant C independent of ε such that

$$\|u^\varepsilon\|_{L^2(\Omega)}\leq C,\quad \|\varepsilon\nabla u^\varepsilon\|_{L^2(\Omega)}\leq C,\quad \|(C^{(1)}(x/\varepsilon))^{1/2}\nabla u^\varepsilon\|_{L^2(\Omega)}\leq C. \tag{4.3.2}$$

Note that here for $\alpha=0$ we require the existence of the following Poincaré type inequality: There exists a constant $C>0$ independent of ε such that for all

$u \in H_0^1(\Omega)$

$$\|u\|^2_{L^2(\Omega)} \le C\left(\|C^{(1)}(x/\varepsilon)\nabla u\|^2_{L^2(\Omega_1^\varepsilon)} + \varepsilon^2\|C^{(0)}(x/\varepsilon)\nabla u\|^2_{L^2(\Omega_0^\varepsilon)}\right),$$

which indeed holds, the proof of this is given in Section 4.4, Lemma 4.4.4. The uniform bounds (4.3.2), and the two-scale compactness result, see Lemma B.0.2 (i), imply that these sequences have two-scale convergent subsequences and the behaviour of their two-scale limits is described in the following Lemma.

Lemma 4.3.1. *There exists $u(x) \in [H_0^1(\Omega)]^d$, $v(x,y) \in [L^2(\Omega; H_0^1(Q_0) \cap V)]^d$ such that, up to a subsequence in ε (which we do not relabel),*

$$\begin{aligned} u^\varepsilon &\overset{2}{\rightharpoonup} u(x) + v(x,y), \\ \varepsilon\nabla u^\varepsilon &\overset{2}{\rightharpoonup} \nabla_y v(x,y), \\ (C^{(1)})^{1/2}(x/\varepsilon)\nabla u^\varepsilon &\overset{2}{\rightharpoonup} (C^{(1)})^{1/2}(y)\left[\nabla_x u(x) + \nabla_y u_1(x,y)\right], \end{aligned}$$

where $u_1(x,y) \in [L^2(\Omega; H^1_\#(Q))]^d$ is a solution to

$$-\mathrm{div}_y\left(C^{(1)}(y)\nabla_y u_1(x,y)\right) = \mathrm{div}_y\left(C^{(1)}(y)\nabla_x u(x)\right) \tag{4.3.3}$$

Here $V \subset [L^2_\#(Q)]^d$ is the space of microscopic oscillations, cf. (4.2.1),

$$V = \{v \in [H^1_\#(Q)]^d : v(y) = k + \chi_0(y)w(y) \text{ for some } k \in \mathbb{R}^d,\ w \in [H_0^1(Q_0)]^d, \\ \mathrm{div}_y w = 0\}.$$

and $W \subset [L^2(Q)]^{d\times d}$ is the space of generalised fluxes, cf. (3.1.12),

$$W := \left\{\phi \in [L^2_\#(Q)]^{d\times d} : \mathrm{div}\left((C^{(1)})^{1/2}\phi\right) = 0 \text{ in } [H^{-1}_\#(Q)]^d\right\}. \tag{4.3.4}$$

Proof of Lemma 4.3.1. By (4.3.2) and the two-scale compactness result, up to a subsequence in ε, which we do not relabel, u^ε, $\varepsilon\nabla u^\varepsilon$ and $(C^{(1)}(x/\varepsilon))^{1/2}u^\varepsilon$ two-scale converge to u^0, $\nabla_y u^0$, and ξ^0 respectively, cf Lemma 3.1.1.

Let us show $u^0 \in [L^2(\Omega; V)]^d$. For appropriate test functions $\phi(x), \psi(y)$, since

$\varepsilon\nabla u^\varepsilon \xrightarrow{2} \nabla_y u^0$, by definition of two-scale convergence

$$\int_\Omega \varepsilon\Big(C^{(1)}\left(\tfrac{x}{\varepsilon}\right)\Big)^{1/2}\nabla u^\varepsilon(x)\cdot\phi(x)\psi(x/\varepsilon)\,\mathrm{d}x = \int_\Omega \varepsilon\nabla u^\varepsilon(x)\cdot\Big(C^{(1)}\left(\tfrac{x}{\varepsilon}\right)\Big)^{1/2}\phi(x)\psi(x/\varepsilon)\,\mathrm{d}x$$
$$\longrightarrow \int_\Omega\int_Q \nabla_y u_0(x,y)\cdot(C^{(1)}(y))^{1/2}\phi(x)\psi(y)\,\mathrm{d}x\mathrm{d}y$$
$$= \int_\Omega\int_Q (C^{(1)}(y))^{1/2}\nabla_y u_0(x,y)\cdot\phi(x)\psi(y)\,\mathrm{d}x\mathrm{d}y,$$

but we know, by (4.3.2), $(C^{(1)}(x/\varepsilon))^{1/2}\nabla u^\varepsilon(x)$ is bounded in $L^2(\Omega)$ so

$$\int_\Omega \varepsilon\Big(C^{(1)}\left(\tfrac{x}{\varepsilon}\right)\Big)^{1/2}\nabla u^\varepsilon(x)\cdot\phi(x)\psi(x/\varepsilon)\,\mathrm{d}x \longrightarrow 0.$$

Therefore
$$\int_\Omega\int_Q (C^{(1)}(y))^{1/2}\nabla_y u_0(x,y)\cdot\phi(x)\psi(y)\,\mathrm{d}x\mathrm{d}y = 0.$$

This implies, since the span of functions of the form $\phi(x)\psi(y)$ is dense in $L^2(\Omega\times Q)$,

$$(C^{(1)}(y))^{1/2}\nabla_y u_0(x,y) = 0 \quad \text{a.e. } x\in\Omega. \tag{4.3.5}$$

Premultiplying (4.3.5) by $(C^{(1)}(y))^{1/2}$ shows $u_0(x,y)\in L^2(\Omega;V)$. Hence, we have shown

$$u^\varepsilon \xrightarrow{2} u_0(x,y),$$
$$\varepsilon\nabla u^\varepsilon \xrightarrow{2} \nabla_y u_0(x,y),$$
$$(C^1(x/\varepsilon))^{1/2}\nabla u^\varepsilon \xrightarrow{2} \xi_0(x,y).$$

For $\Psi\in C^\infty(\Omega;W)$ we see via integration by parts

$$\int_\Omega \Big(C^{(1)}\left(\tfrac{x}{\varepsilon}\right)\Big)^{1/2}\nabla u^\varepsilon(x)\cdot\Psi(x,x/\varepsilon)\,\mathrm{d}x = -\int_\Omega u^\varepsilon\cdot\left(\operatorname{div}_x\Big(C^{(1)}\left(\tfrac{x}{\varepsilon}\right)\Big)^{1/2}\Psi\right)$$

and passing to the two-scale limit indicates $u^0(x,y)$ and $\xi^0(x,y)$ are related by the following expression

$$\int_\Omega \int_Q \xi_0(x,y)\cdot\Psi(x,y)\ \mathrm{d}x\mathrm{d}y = -\int_\Omega\int_Q u_0(x,y)\cdot \mathrm{div}_x\left((C^{(1)}(y))^{1/2}\Psi(x,y)\right)\ \mathrm{d}x\mathrm{d}y \quad \forall \Psi \in C^\infty(\Omega; W). \tag{4.3.6}$$

For a.e. $x\in\Omega$, $u_0(x,\cdot)\in V$, and by (4.2.1) we see $u_0(x,y) = u(x) + v(x,y)$ for some $u\in[L^2(\Omega)]^d$, $v\in[L^2(\Omega;H^1_0(Q_0))]^d$ with $\mathrm{div}_y v = 0$. Let us now show that $u(x)\in[H^1_0(\Omega)]^d$. By taking in (4.3.6) test functions of the form $\Psi\in C^\infty(\Omega;W)$ such that $\Psi = 0$ on Q_0 we arrive at

$$\int_\Omega \int_Q \xi_0(x,y)\cdot\Psi(x,y)\ \mathrm{d}x\mathrm{d}y = -\int_\Omega\int_Q u(x)\cdot \mathrm{div}_x\left((C^{(1)})^{1/2}(y)\Psi(x,y)\right)\ \mathrm{d}x\mathrm{d}y. \tag{4.3.7}$$

Now, we construct such a test function Ψ in a similar fashion as in Section 3.2.2. That is, let Ψ be given by (3.2.11) constructed using (3.2.10) with a^{hom} replaced by C^{hom}, the perforated domain homogenised elasticity tensor:

$$C^{\mathrm{hom}}_{ijrs} = \int_{Q_1} C^{(1)}_{ijpq}(y)\left(\delta_{pr}\delta_{qs} + \frac{\partial N^p_{rs}}{\partial y_q}\right)\mathrm{d}y,$$

for N_{rs} the solutions of the perforated domain cell problem. We see, see Section 3.2.2, that (4.3.7) implies $u\in[H^1_0(\Omega)]^d$.

Let us now show $\xi_0(x,y) = (C^{(1)})^{1/2}(y)[\nabla_x u(x) + \nabla_y u_1(x,y)]$ for some u_1 given by (4.3.3). For a.e. $x\in\Omega$, let $u_1(x,\cdot)\in[H^1_\#(Q)]^d$ be a solution of

$$-\mathrm{div}_y\left(C^{(1)}(y)\nabla_y u_1(x,y)\right) = \mathrm{div}_y\left(C^{(1)}(y)\nabla_x u(x)\right)$$

Note that such a solution exists by Lemma 3.1.4 since, for $F := \mathrm{div}_y\left(C^{(1)}(y)\nabla_x u(x)\right)$,

$$< F, v >= \int_Q C^{(1)}(y)\nabla_x u(x)\cdot\nabla_y v\ \mathrm{d}y = \int_Q \nabla_x u(x)\cdot C^{(1)}(y)\nabla_y v\ \mathrm{d}y = 0 \quad \forall v\in V.$$

Setting $\xi(x,y) := (C^{(1)})^{1/2}[\nabla_x u(x) + \nabla_y u_1(x,y)]$, we note ξ is unique, since $u_1(x,\cdot)$

is unique up to a function in V. Furthermore, $\xi(x,y) \in [L^2(\Omega; W)]^d$ with

$$\int_\Omega \int_Q \xi(x,y) \cdot \Psi(x,y) \; \mathrm{d}x\mathrm{d}y = -\int_\Omega \int_Q u(x) \cdot \mathrm{div}_x \left((C^{(1)}(y))^{1/2} \Psi(x,y) \right) \; \mathrm{d}x\mathrm{d}y \qquad \forall \Psi \in C^\infty(\Omega; W). \tag{4.3.8}$$

It remains to show $\xi_0(x,y) = \xi(x,y)$ for a.e. $x \in \Omega$: since $\xi_0(x,\cdot)$ and $\xi(x,\cdot)$ belong to W, it is sufficient to show $\xi_0(x,\cdot) - \xi(x,\cdot) \perp W$. To this end, (4.3.6) and (4.3.8) imply that

$$\int_\Omega \int_Q (\xi_0(x,y) - \xi(x,y)) \cdot \Psi(x,y) \; \mathrm{d}x\mathrm{d}y = \\ -\int_\Omega \int_Q v(x,y) \cdot \mathrm{div}_x \left((C^{(1)}(y))^{1/2} \Psi(x,y) \right) \mathrm{d}x\mathrm{d}y \quad \forall \Psi \in C^\infty(\Omega; W). \tag{4.3.9}$$

Now the result follows if the right hand side of (4.3.9) is zero. Since $C_0^\infty(\Omega; V)$ is dense in $L^2(\Omega; V)$ it is sufficient to show: for fixed $\phi \in [C_0^\infty(\Omega; V)]^d$

$$\int_\Omega \int_{Q_0} (C^{(1)}(y))^{1/2} \nabla_x \phi(x,y) \cdot \Psi \; \mathrm{d}x\mathrm{d}y = 0 \quad \forall \Psi \in C^\infty(\Omega; W) \tag{4.3.10}$$

This can be seen by considering the function $N \in [H^1_\#(Q)]^d$ such that $N(y) = y$ for $y \in Q_0$. An example of such a function would be an appropriate mollification of the function $f(y) = \chi_0(y) y$, which is possible since Q_0 is strictly included in Q. Now for given ϕ define an auxiliary function $\Phi(x,y) := \nabla_x \phi(x,y) \cdot N(y)$. For a.e. $x \in \Omega$, $\Phi(x,\cdot) \in [H^1_\#(Q)]^d$, and for any $\Psi \in W$

$$\begin{aligned} 0 = \langle -\mathrm{div}_y (C^{(1)}(y))^{1/2} \Psi), \Phi(x,y) \rangle &= \int_Q \Psi(y) \cdot (C^{(1)}(y))^{1/2} \nabla_y \Phi(x,y) \; \mathrm{d}y \\ &= \int_{Q_0} \Psi(y) \cdot (C^{(1)}(y))^{1/2} \nabla_y (\nabla_x \phi(x,y) \cdot y) \; \mathrm{d}y \\ &= \int_{Q_0} \Psi(y) \cdot (C^{(1)}(y))^{1/2} \nabla_x \phi(x,y) \; \mathrm{d}y, \end{aligned}$$

where the last equality is due to the fact that, for $\phi(x,\cdot) \in V$,

$$
\begin{aligned}
((C^{(1)}(y))^{1/2}\nabla_y(\nabla_x\phi(x,y)\cdot y))_{ij} &= (C^{(1)}(y))^{1/2}_{ijpq}(\phi_{p,x_s}y_s)_{,y_q} \\
&= (C^{(1)}(y))^{1/2}_{ijpq}(\phi_{p,x_sy_q}y_s + \phi_{p,x_s}\delta_{sq}) \\
&= \underbrace{(C^{(1)}(y))^{1/2}_{ijpq}(\phi_{p,x_sy_q}y_s)}_{=0} + (C^{(1)}(y))^{1/2}_{ijpq}(\phi_{p,x_q}) \\
&= ((C^{(1)}(y))^{1/2}\nabla_x\phi(x,y))_{ij}
\end{aligned}
$$

□

Using Lemma 4.3.1 we are now able to pass to the limit as $\varepsilon \to 0$ in (4.3.1) and find the homogenised equation for $u^0(x,y)$.

Theorem 4.3.2. *Let $f^\varepsilon(x)$ weakly (strongly) two-scale converge to $f(x,y)$ as $\varepsilon \to 0$. Then the sequence u^ε weakly (strongly) two-scale converges to $u^0(x,y) = u(x) + v(x,y)$ as $\varepsilon \to 0$, where $(u,v) \in [H^1_0]^d \times [L^2(\Omega; H^1_0(Q_0))]^d$ is the unique solution to*

$$-\mathrm{div}\left(C^{\mathrm{hom}}\nabla u(x)\right) + \alpha u(x) + \alpha < v > (x) =< f > (x) \quad \text{in } \Omega, \tag{4.3.11}$$

$$
\begin{aligned}
-\Delta_y v(x,y) + \alpha v(x,y) &= f(x,y) + \nabla p(x,y) \quad \text{in } Q_0 \\
\mathrm{div}_y v(x,y) &= 0 \quad \text{in } Q_0 \qquad (4.3.12) \\
v(x,y) &= 0 \quad \text{on } \partial Q_0,
\end{aligned}
$$

where $p \in H^1(Q_0)$ is also unknown. C^{hom} is the constant coefficient positive tensor given by

$$C^{\mathrm{hom}}_{ijrs} = \int_Q C^{(1)}_{ijpq}(y)\left(\delta_{pr}\delta_{qs} + \frac{\partial N^p_{rs}}{\partial y_q}\right)\mathrm{d}y. \tag{4.3.13}$$

Here $N_{rs} = (N^1_{rs}, N^2_{rs}, \ldots, N^d_{rs})$ is a Q-periodic solution to the degenerate cell problem

$$-\mathrm{div}_y\left(C^{(1)}(y)\left(e_r \otimes e_s + \nabla_y N_{rs}(y)\right)\right) = 0 \quad \text{in } Q. \tag{4.3.14}$$

Proof of Theorem 4.3.2. For fixed $\phi(x,y) \in [C^\infty_0(\Omega;V)]^d$, substituting the test function $\phi^0(x) := \phi(x, \frac{x}{\varepsilon})$ into (4.3.1) gives

$$\int_\Omega C^{(1)}(\tfrac{x}{\varepsilon})\nabla u^\varepsilon \cdot \nabla_x \phi(x, \tfrac{x}{\varepsilon}) + \varepsilon C^{(0)}(\tfrac{x}{\varepsilon})\nabla u^\varepsilon \cdot \left[\varepsilon \nabla_x \phi(x, \tfrac{x}{\varepsilon}) + \nabla_y \phi(x, \tfrac{x}{\varepsilon})\right] + \\ + \int_\Omega \alpha u^\varepsilon \cdot \phi(x, \tfrac{x}{\varepsilon}) = \int_\Omega f^\varepsilon \cdot \phi(x, \tfrac{x}{\varepsilon}). \quad (4.3.15)$$

Passing to the limit $\varepsilon \to 0$, using Lemma 4.3.1, gives

$$\int_\Omega \int_Q C^{(1)}(y) \left[\nabla_x u(x) + \nabla_y u^1(x,y)\right] \cdot \nabla_x \phi(x,y) \ \mathrm{d}y\mathrm{d}x + \\ + \int_\Omega \int_Q C^{(0)}(y) \nabla_y v(x,y) \cdot \nabla_y \phi(x,y) + \alpha \left(u(x) + v(x,y)\right) \cdot \phi(x,y) \ \mathrm{d}y\mathrm{d}x \\ = \int_\Omega \int_Q f(x,y)\phi(x,y) \ \mathrm{d}y\mathrm{d}x, \quad \forall \phi(x,y) \in [C_0^\infty(\Omega; V)]^d. \quad (4.3.16)$$

Choosing $\phi(x,y) \equiv \varphi(x)$, $\varphi \in C_0^\infty(\Omega)$, in (4.3.16) gives

$$\int_\Omega \int_Q C^{(1)}(y)(\nabla_x u(x) + \nabla_y u^1(x,y)) \cdot \nabla_x \varphi(x) + \alpha \left(u(x) + v(x,y)\right) \varphi(x) \ \mathrm{d}x\mathrm{d}y \\ = \int_\Omega \int_Q f(x,y)\varphi(x) \ \mathrm{d}x\mathrm{d}y,$$

which is the variational form for

$$-\mathrm{div} \left(\left\langle C^{(1)}(\nabla_x u + \nabla_y u^1)\right\rangle (x)\right) + \alpha \left(u(x) + < v > (x)\right) = < f > (x) \quad \text{in } \Omega. \quad (4.3.17)$$

Setting $u_p^1 = N_{rs}^p(y)\frac{\partial u_r}{\partial x_s}(x)$ and substituting into (4.3.3) and (4.3.17) gives equations (4.3.11) and (4.3.13)-(4.3.14). Let us show C^{hom} is strictly positive. For any symmetric $\eta \in \mathbb{R}^{d\times d}$ we can see $C^{\mathrm{hom}}\eta \cdot \eta$ has the following variational representation

$$C^{\mathrm{hom}}\eta \cdot \eta = \inf_{w \in [C_\#^\infty(Q)]^d} \int_Q C^{(1)}(y) \left(\eta + \nabla_y w\right) \left(\eta + \nabla_y w\right) \ \mathrm{d}y.$$

By (4.1.2) it is easy to see,

$$
\begin{aligned}
C^{\text{hom}}\eta\cdot\eta &\geq \inf_{w\in[C^\infty_\#(Q)]^d}\int_Q C^{(2)}(y)\left(\eta+\nabla_y w\right)\left(\eta+\nabla_y w\right)\,\mathrm{d}y\\
&= \hat{C}\eta\cdot\eta,
\end{aligned}
$$

where $\hat{C}$ is the homogenised tensor for the high contrast linear elasticity problem which is well known to be strictly positive, cf. for example [39]. Therefore C^{hom} is strictly positive.

For $\phi_0(x,y)=\psi(x)\phi(y)$, $\psi\in C_0^\infty(\Omega)$, $\phi\in C_0^\infty(Q_0)$ with $\mathrm{div}\phi=0$, we have

$$
\int_\Omega\int_Q (C^{(1)}(y))\left[\nabla_x u(x)+\nabla_y u^1(x,y)\right]\nabla_x\psi(x)\phi(y)\,\mathrm{d}x\mathrm{d}y=0
$$

due to (4.3.10) and the fact $\xi_0(x,y)=\left(C^{(1)}(y)\right)^{1/2}\left[\nabla_x u(x)+\nabla_y u^1(x,y)\right]$ $\in[L^2(\Omega;W)]^d$. Furthermore

$$
\begin{aligned}
\int_\Omega\int_{Q_0} C^{(0)}(y)\nabla_y v(x,y)\cdot\nabla_y\phi(y)\,\mathrm{d}x\mathrm{d}y &= \int_\Omega\int_{Q_0}(\delta_{ip}\delta_{jq}+\delta_{iq}\delta_{jp})v_{p,y_q}\phi_{i,y_j}\,\mathrm{d}x\mathrm{d}y\\
= \int_\Omega\int_{Q_0}(v_{p,y_q}\phi_{p,y_q}+v_{p,y_q}\phi_{q,y_p})\,\mathrm{d}x\mathrm{d}y &= \int_\Omega\int_{Q_0}(v_{p,y_q}\phi_{p,y_q}+v_{p,y_p}\phi_{q,y_q})\,\mathrm{d}x\mathrm{d}y\\
&= \int_\Omega\int_{Q_0}\nabla_y v\cdot\nabla_y\phi\,\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Therefore choosing test functions in (4.3.16) to be of the form $\phi_0(x,y)=\psi(x)\phi(y)$, $\psi\in C_0^\infty(\Omega)$, $\phi\in C_0^\infty(Q_0)$ with $\mathrm{div}\phi=0$ gives

$$
\begin{aligned}
\int_\Omega\int_{Q_0}\nabla_y v(x,y)\cdot\nabla_y\phi(x)\psi(y)+\alpha\left(u(x)+v(x,y)\right)\cdot\psi(x)\phi(y)\,\mathrm{d}x\mathrm{d}y&\\
=\int_\Omega\int_{Q_0} f(x,y)\psi(x)\phi(y)\,\mathrm{d}x\mathrm{d}y&,
\end{aligned}
$$

which is the variational form for (4.3.12).

It remains to show if $f^\varepsilon\overset{2}{\rightarrow}f$ then $u^\varepsilon\overset{2}{\rightarrow}u^0$. This proof is similar to the double porosity case, see [37], and we will present it here for completeness. Let $z^\varepsilon\in[H_0^1(\Omega)]^d$ be the solution to

$$
-\mathrm{div}\left(C^\varepsilon\nabla z^\varepsilon\right)+\alpha z^\varepsilon=u^\varepsilon. \tag{4.3.18}
$$

Then, by the above arguments, $z^\varepsilon \stackrel{2}{\rightharpoonup} z^0$ where $z^0(x,y) = z(x) + w(y)$ is the solution to

$$\int_\Omega \int_Q C^{(1)}_{ijpq}(y) \left[\delta_{pq}\delta_{rs} + N^p_{rs,q}(y)\right] z_{r,s}(x)\phi_{i,x_j}(x,y)\ \mathrm{d}y\mathrm{d}x +$$
$$+ \int_\Omega \int_Q C^{(0)}(y)\nabla_y w(x,y) \cdot \nabla_y \phi(x,y) + \alpha\left(z(x) + w(x,y)\right) \cdot \phi(x,y)\ \mathrm{d}y\mathrm{d}x$$
$$= \int_\Omega \int_Q u^0(x,y)\phi(x,y)\ \mathrm{d}y\mathrm{d}x, \quad \forall \phi(x,y) \in [C_0^\infty(\Omega; C^\infty_\#(Q))]^d. \quad (4.3.19)$$

Setting $\phi = z^0$ in (4.3.16) and $\phi = u^0$ in (4.3.19) shows

$$\int_\Omega \int_Q \left(u^0(x,y)\right)^2\ \mathrm{d}y\mathrm{d}x = \int_\Omega \int_Q f(x,y) z^0(x,y)\ \mathrm{d}y\mathrm{d}x. \quad (4.3.20)$$

Similarly, the variational forms for (4.3.18) and (4.1.1) show

$$\int_\Omega |u^\varepsilon(x)|^2\ \mathrm{d}x = \int_\Omega f^\varepsilon(x) \cdot z^\varepsilon(x)\ \mathrm{d}x. \quad (4.3.21)$$

Using the assumption $f^\varepsilon \stackrel{2}{\to} f^0$, passing to the limit $\varepsilon \to 0$ in (4.3.21), via (4.3.20), gives

$$\lim_{\varepsilon\to 0} \int_\Omega |u^\varepsilon|^2\ \mathrm{d}x = \lim_{\varepsilon \to 0} \int_\Omega f^\varepsilon(x) \cdot z^\varepsilon(x)\ \mathrm{d}x = \int_\Omega \int_Q f(x,y) \cdot z^0(x,y)\ \mathrm{d}y\mathrm{d}x$$
$$= \int_\Omega \int_Q |u^0(x,y)|^2\ \mathrm{d}y\mathrm{d}x.$$

Hence, via Lemma B.0.4, $u^\varepsilon \stackrel{2}{\to} u^0$. □

Proof of Theorem 4.1.1. The equation for the microscopic deformations $v(x,y)$ is given by

$$\int_\Omega \int_{Q_0} \nabla_y v(x,y) \cdot \nabla_y \psi(y)\phi(x) + \alpha\Big(u(x) + v(x,y)\Big) \cdot \phi(x)\psi(y)\ \mathrm{d}x\mathrm{d}y$$
$$= \int_\Omega \int_{Q_0} f(x,y)\phi(x)\psi(y)\ \mathrm{d}x\mathrm{d}y, \quad (4.3.22)$$

for all $\phi \in C_0^\infty(\Omega)$, $\psi \in C_0^\infty(Q_0)$ with $\mathrm{div}\psi = 0$. For an externally applied body force $f^\varepsilon(x) = f(x, x/\varepsilon)$, where $f(x,y) = f_0(x) + \nabla_y f_1(x,y)$, such that

$f^\varepsilon \overset{2}{\rightharpoonup} f(x,y)$ as $\varepsilon \to 0$ we find, by taking into account that for any constant vector field c we can have the representation $c = \nabla_y(c \cdot y)$ in $L^2(Q_0)$, that

$$\begin{aligned}\int_\Omega \int_{Q_0} f(x,y) \cdot \phi(x)\psi(y) \, \mathrm{d}x\mathrm{d}y &= \int_\Omega \int_{Q_0} \left[f_0(x) + \nabla_y f_1(x,y)\right] \phi(x)\psi(y) \, \mathrm{d}x\mathrm{d}y \\ &= \int_\Omega \int_{Q_0} \left[\nabla_y \left(f_0(x) \cdot y\right) + \nabla_y f_1(x,y)\right] \phi(x)\psi(y) \, \mathrm{d}x\mathrm{d}y \\ &= -\int_\Omega \int_{Q_0} \left[f_0(x) \cdot y + f_1(x,y)\right] \phi(x) \mathrm{div}_y \psi(y) \, \mathrm{d}x\mathrm{d}y \\ &= 0,\end{aligned}$$

for all $\phi \in C_0^\infty(Q_0)$ with $\mathrm{div}\phi = 0$. Therefore equation (4.3.22) becomes

$$\int_\Omega \int_{Q_0} \nabla_y v(x,y) \cdot \nabla_y \psi(y)\phi(x) + \alpha\Big(u(x) + v(x,y)\Big) \cdot \phi(x)\psi(y) \, \mathrm{d}x\mathrm{d}y = 0.$$

This implies that for a.e. x in Ω, $v(x,\cdot)$ is a weak solution of the homogeneous Stokes problem which is well known to have only the trivial solution $v(x,y) = 0$, see e.g. [36]. Furthermore, if $u^\varepsilon \overset{2}{\rightharpoonup} u(x)$ then by properties of two-scale convergence, see Appendix B, $u^\varepsilon \rightharpoonup u$ in L^2. Moreover, u^ε is a bounded sequence in $H_0^1(\Omega)$ which implies that u^ε converges weakly to some $\hat{u} \in H_0^1(\Omega)$. Hence, by the compact embedding $L^2(\Omega) \subset H^1(\Omega)$, $u^\varepsilon \to \hat{u}$ strongly in L^2 and by uniqueness of the weak limit $\hat{u} = u$. This is Theorem 4.1.1. $\square$

4.4 Spectral Compactness

4.4.1 Spectrum of the two-scale homogenised limit operator

Let us study the eigenvalues of the homogenised limit operator A^0 given by Theorem 4.1.2. That is we wish to find for which λ there exists a nontrivial

$u^0(x,y) = u(x) + v(x,y)$, $u(x) \in [H_0^1(\Omega)]^d$, $v(x,y) \in [H_0^1(Q_0)]^d$ that solve

$$-\text{div}\left(C^{\text{hom}}\nabla u(x)\right) = \lambda u(x) + \lambda < v > (x) \quad \text{in } \Omega, \tag{4.4.1}$$

$$\begin{aligned} -\Delta_y v(x,y) &= \lambda v(x,y) + \lambda u(x) + \nabla_y p(x,y) \quad \text{in } Q_0 \\ \text{div}_y v(x,y) &= 0 \quad \text{in } Q_0 \\ v(x,y) &= 0 \quad \text{on } \partial Q_0. \end{aligned} \tag{4.4.2}$$

We notice, by Theorem 4.1.1, that the $\lambda u(x)$ term in (4.4.2) can be removed by an appropriate choice of p. Therefore, studying (4.4.1)-(4.4.2) is equivalent to studying

$$-\text{div}\left(C^{\text{hom}}\nabla u(x)\right) = \lambda u(x) + \lambda < v > (x) \quad \text{in } \Omega, \tag{4.4.3}$$

$$\begin{aligned} -\Delta_y v(x,y) &= \lambda v(x,y) + \nabla_y p(x,y) \quad \text{in } Q_0 \\ \text{div}_y v(x,y) &= 0 \quad \text{in } Q_0 \\ v(x,y) &= 0 \quad \text{on } \partial Q_0, \end{aligned} \tag{4.4.4}$$

for some unknown $p \in L^2(\Omega; H^1(Q_0))$.

It is well known that, for $n = 1, 2, \ldots$, there exist eigenvalues λ_n^D such that

$$0 < \lambda_1^D \leq \lambda_2^D \leq \lambda_3^D \leq \ldots$$

and eigenfunctions $u_n \in [H_0^1(\Omega)]^d$ such that

$$-\text{div}\left(C^{\text{hom}}\nabla u_n(x)\right) = \lambda_n^D u_n(x) \quad \text{in } \Omega.$$

By setting $(v,p) \equiv (0,0)$ we see that λ_n^D is in the spectrum of A^0 with corresponding eigenfunction $u_n^0(x,y) = u_n(x)$.

It is also well known, for the Stokes spectral problem (4.4.4), that for $n = 1, 2, \ldots$, there exist μ_n such that

$$0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \ldots$$

and $v_n \in [H_0^1(Q_0)]^d$, $p_n \in H^1(Q_0)$ such that

$$\begin{aligned} -\Delta_y v_n(y) &= \mu_n v_n(y) + \nabla_y p_n(y) && \text{in } Q_0 \\ \operatorname{div}_y v_n(y) &= 0 && \text{in } Q_0. \end{aligned}$$

If, for some $N = 1, 2, \ldots$, $\langle v_N \rangle = 0$ then μ_N is clearly in the spectrum of A^0 with corresponding eigenfunction $u_N^0(x,y) = v_N(y)$. For the eigenvalues μ_n whose corresponding eigenfunctions have non-zero mean, i.e. $\langle v_n \rangle \neq 0$, assuming $\mu_n \neq \lambda_m^D$ for all m let $u_n(x) \in [H_0^1(\Omega)]^d$ be the solution of

$$-\operatorname{div}\left(C^{\text{hom}} \nabla u_n(x)\right) = \mu_n u_n(x) + \mu_n < v_n > (x) \quad \text{in } \Omega.$$

Then μ_n is in the spectrum of A^0 with corresponding eigenfunction $u_n^0(x,y) = u_n(x) + v_n(x,y)$. If $\mu_n = \lambda_m$ for some m we have already shown above that μ_n will lie in the spectrum of A^0.

It remains to show that we have exhausted all possible eigenvalues of the spectrum of A^0. That is the following result holds.

Lemma 4.4.1 (Spectrum of the limit operator)**.** *The spectrum of the homogenised limit operator A^0, $\sigma(A^0)$, has the following representation:*

$$\sigma(A^0) = \{\lambda_n \mid n \in \mathbb{N}\} \cup \{\mu_n \mid n \in \mathbb{N}\}.$$

Here λ_n satisfies, for some non-trivial $u_n \in [H_0^1(\Omega)]^d$,

$$-\operatorname{div}\left(C^{\text{hom}} \nabla u_n(x)\right) = \lambda_n u_n(x) \quad \textit{in } \Omega,$$

and μ_n satisfies, for some non-trivial $v_n \in [H_0^1(Q_0)]^d$, $p_n \in H^1(Q_0)$,

$$\begin{aligned} -\Delta_y v_n(y) &= \mu_n v_n(y) + \nabla_y p_n(y) && \textit{in } Q_0 \\ \operatorname{div}_y v_n(y) &= 0 && \textit{in } Q_0. \end{aligned}$$

Proof. For λ_n, μ_n given in Lemma 4.4.1 we showed in the above discussion

$$\{\lambda_n \mid n \in \mathbb{N}\} \cup \{\mu_n \mid n \in \mathbb{N}\} \subset \sigma(A^0).$$

To show the reverse inclusion it is sufficient to show that if $\lambda \neq \lambda_n$, $\lambda \neq \mu_m$, $\forall n$, $\forall m$ then for a given $f(x,y) \in [L^2(\Omega \times Q)]^d$ there exists a unique solution $u = u_0(x) + v(x,y)$, $u_0 \in [H_0^1(\Omega)]^d$, $v \in [L^2(\Omega; H_0^1(Q_0))]^d$, continuously depending on f to

$$-\mathrm{div}_x(C^{\mathrm{hom}}\nabla_x u(x)) - \lambda u(x) = \lambda < v > (x) + < f > (x), \tag{4.4.5}$$

and

$$\begin{aligned} -\Delta_y v(x,y) - \lambda v(x,y) &= f(x,y) + \nabla_y p(x,y), \\ -\mathrm{div}_y v(x,y) &= 0, \end{aligned} \tag{4.4.6}$$

for some $p(x,y) \in [L^2(\Omega; H^1(Q_0))]^d$. It is well known that if $\lambda \neq \mu_m$, $\forall m$, then there exists a unique v to problem (4.4.6). Furthermore, $g := \lambda < v > + < f > \in [L^2(\Omega)]^d$ and it is well known if $\lambda \notin \lambda_n$, $\forall n$, there exists a unique $u_0 \in [H_0^1(\Omega)]^d$ such that

$$-\mathrm{div}_x(C^{\mathrm{hom}}\nabla_x u(x)) - \lambda u(x) = g(x).$$

The above construction ensures, by the boundedness of the appropriate inverse operators, the continuity of the solution in f. □

4.4.2 Convergence of spectra

Lemma 4.4.2 (Spectral compactness Lemma)**.** *The spectrum of A^ε, $\sigma(A^\varepsilon)$, converges in the sense of Hausdorff to the spectrum of A^0, $\sigma(A^0)$. That is*

(i) For every $\lambda \in \sigma(A^0)$ there exists $\lambda^\varepsilon \in \sigma(A^\varepsilon)$ such that $\lambda^\varepsilon \to \lambda$.

(ii) If there exists $\lambda^\varepsilon \in \sigma(A^\varepsilon)$ such that $\lambda^\varepsilon \to \lambda$, then $\lambda \in \sigma(A^0)$.

According to Appendix B, property (i) is implied by the convergence, in the strong two-scale resolvent sense, of the operator A^ε to A, which was proven in Theorem 4.1.2. For property (ii), we prove below by arguments that are conceptually similar to [37]. Assuming $\lambda^\varepsilon \to \lambda^0$, let u^ε be the corresponding normalised eigenfunctions of λ^ε, i.e.

$$A^\varepsilon u^\varepsilon = \lambda^\varepsilon u^\varepsilon, \qquad \|u^\varepsilon\|_{L^2(\Omega)} = 1. \tag{4.4.7}$$

Since the sequence u^ε is bounded, $u^\varepsilon \xrightharpoonup{2} u^0$ for some $u^0 \in L^2(\Omega \times Q)$. By (4.4.7) we can see:

$$A^\varepsilon u^\varepsilon + u^\varepsilon = (\lambda^\varepsilon + 1)u^\varepsilon =: f^\varepsilon,$$

and, by Theorem (4.1.2), passing to the limit $\varepsilon \to 0$ gives

$$A^0 u^0 + u^0 = (\lambda^0 + 1)u^0.$$

That is,

$$A^0 u^0 = \lambda^0 u^0,$$

for some u^0. To assure λ^0 is in the spectrum of A^0 is to show u^0 is not identically zero.

The quadratic form

$$\mathcal{B}(u, v) := \int_{Q_0} \nabla_y u(y) \cdot \nabla_y v(y) \, \mathrm{d}y$$

on the domain $H := \{v \in H_0^1(Q_0) : \mathrm{div} v = 0\}$ defines a self-adjoint operator B. It is well known the operator B has a compact resolvent and therefore a discrete spectrum of eigenvalues going to infinity. Furthermore, by Lemma 4.4.1 $\sigma(B) = \{\mu_n : n \in \mathbb{N}\} \subset \sigma(A^0)$. To prove $u^0 \neq 0$ it is sufficient to show the following result

Lemma 4.4.3. *Suppose that*

$$A^\varepsilon u^\varepsilon = \lambda^\varepsilon u^\varepsilon, \qquad \qquad \|u^\varepsilon\|_{L^2(\Omega)} = 1.$$

Let $\lambda^\varepsilon \to \lambda \notin \sigma(B)$. *Then* u_ε *has a strongly two-scale convergent subsequence.*

Indeed, if Lemma 4.4.3 holds then by the properties of strong two-scale convergence, Lemma (B.0.4) in Appendix B, $\|u^\varepsilon\|_{L^2(\Omega)} \to \|u^0\|_{L^2(\Omega\times Q)}$ which implies $u^0 \neq 0$.

Proof. The Modification Lemma 4.2.1 can be easily extended to functions $u \in [H^1(Q)]^d$ if the inequality in Lemma 4.2.1 (*iii*) is replaced by a slightly weaker inequality:

$$\|\hat{v}\|^2_{H^1(Q)} \le c \left(\|v\|_{L^2(Q_1)} + \|e(v)\|^2_{L^2(Q_1)} + \|\nabla \cdot u\|^2_{L^2(Q_0)} \right). \tag{4.4.8}$$

This can be seen by following the proof of Lemma 4.2.1 and using, where appropriate, the Korn's inequality C.0.6 instead of the periodic Korn's inequality C.0.7. Since, by assumption, the inclusion phase Ω_0^ε does not intersect the boundary $\partial\Omega$, we can use the Modification lemma to construct $\hat{u}^\varepsilon$ a modification of u^ε such that $\hat{u}^\varepsilon = u^\varepsilon$ in Ω_1^ε; $\mathrm{div}\hat{u}^\varepsilon = \mathrm{div}u^\varepsilon$ in Ω_0^ε and $\Delta\hat{u}^\varepsilon = \nabla\phi^\varepsilon$ for some $\phi^\varepsilon \in L^2(\Omega_0^\varepsilon)$. By a straightforward rescaling of (4.4.8), (4.3.2) and the fact that $\|u^\varepsilon\|_{L^2(\Omega)} = 1$, we find $\|\hat{u}^\varepsilon\|_{H^1(\Omega)} \leq C$. Therefore up to a subsequence $\hat{u}^\varepsilon$ convergences strongly in $L^2(\Omega)$ to some $\hat{u}$. To prove the result it remains to show the difference $v^\varepsilon = u^\varepsilon - \hat{u}^\varepsilon$ strongly two-scale converges.

By construction $v^\varepsilon \in [H_0^1(\Omega_0^\varepsilon)]^d$, $\mathrm{div}_y v^\varepsilon = 0$ and since u^ε solves

$$\int_\Omega C^{(1)}\nabla u^\varepsilon \cdot \nabla\phi \;\mathrm{d}x + \varepsilon^2 \int_{\Omega_0^\varepsilon} C^{(0)}\nabla u^\varepsilon \cdot \nabla\phi \;\mathrm{d}x = \int_\Omega \lambda^\varepsilon u^\varepsilon \phi \;\mathrm{d}x \quad \forall \phi \in C_0^\infty(\Omega),$$

we see that v^ε solves

$$\varepsilon^2 \int_{\Omega_0^\varepsilon} \nabla v^\varepsilon \cdot \nabla\phi \;\mathrm{d}x = \int_\Omega \lambda^\varepsilon (v^\varepsilon + \hat{u}^\varepsilon)\phi \;\mathrm{d}x, \quad \forall \phi \in C_0^\infty(\Omega_0^\varepsilon) \text{ such that } \mathrm{div}\phi = 0. \tag{4.4.9}$$

Let us consider the following variational Stokes problem: Find $w^\varepsilon \in [H_0^1(\Omega_0^\varepsilon)]^d$, $\mathrm{div}_y w^\varepsilon = 0$ such that

$$\varepsilon^2 \int_{\Omega_0^\varepsilon} \nabla w^\varepsilon \cdot \nabla\phi \;\mathrm{d}x = \int_\Omega \lambda^\varepsilon w^\varepsilon \phi + f^\varepsilon \phi \;\mathrm{d}x, \quad \forall \phi \in [C_0^\infty(\Omega_0^\varepsilon)]^d \text{ such that } \mathrm{div}\phi = 0, \tag{4.4.10}$$

for given f^ε. It remains to show the following result: If $f^\varepsilon \stackrel{2}{\rightharpoonup} f$ then $w^\varepsilon \stackrel{2}{\rightharpoonup} w$ where $w \in [L^2(\Omega; H)]^d$ solves

$$\begin{aligned}\int_\Omega \int_{Q_0} \nabla_y w(x,y) \cdot \psi(x)\nabla_y\phi(y) \;\mathrm{d}x\mathrm{d}y &= \int_\Omega \int_{Q_0} \lambda w(x,y)\psi(x)\phi(y) \;\mathrm{d}x\mathrm{d}y \\ + \int_\Omega \int_{Q_0} f(x,y)\psi(x)\phi(y) \;\mathrm{d}x\mathrm{d}y, &\quad \forall \psi \in C_0^\infty(\Omega), \phi \in [C_0^\infty(Q_0)]^d, \mathrm{div}_y\phi = 0.\end{aligned} \tag{4.4.11}$$

Indeed, if this is true then

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega f^\varepsilon v^\varepsilon\ \mathrm{d}x &= \lim_{\varepsilon\to 0}\int_\Omega \varepsilon^2\nabla w^\varepsilon\nabla v^\varepsilon - \lambda^\varepsilon w^\varepsilon v^\varepsilon\ \mathrm{d}x = \lim_{\varepsilon\to 0}\int_\Omega \lambda^\varepsilon \hat{u}^\varepsilon w^\varepsilon\ \mathrm{d}x\\ &= \int_\Omega\int_{Q_0} \lambda\hat{u}w\ \mathrm{d}x\mathrm{d}y = \int_\Omega\int_{Q_0} \nabla_y v\cdot\nabla_y w - \lambda v w\ \mathrm{d}x\mathrm{d}y\\ &= \int_\Omega\int_{Q_0} f v\ \mathrm{d}x\mathrm{d}y,\end{aligned}$$

where the first two equalities come from choosing test functions v^ε in (4.4.10) and w^ε in (4.4.9) respectively; the third equality uses the fact that $\hat{u}^\varepsilon \xrightarrow{2} \hat{u}$ and finally we use symmetry of limiting problem (4.4.11). By choosing $f^\varepsilon = v^\varepsilon$ we have shown, in particular,

$$\lim_{\varepsilon\to 0}\int_\Omega (v^\varepsilon(x))^2\ \mathrm{d}x = \int_\Omega\int_{Q_0} v^2(x,y)\ \mathrm{d}x\mathrm{d}y.$$

Hence $v^\varepsilon \xrightarrow{2} v$.

To show the result (4.4.11) we note that since the domain Ω_0^ε consists of disjoint balls, the spectrum of B^ε, the variational Stokes operator defined on the physical domain Ω_0^ε, coincides with the spectrum of B on a single isolated inclusion (by change of variables in (4.4.10)). Therefore, since $\{\lambda^\varepsilon\}$ is a bounded sequence and $\lambda^\varepsilon \notin \sigma(A_0)$ for small enough ε,

$$\|(B^\varepsilon - \lambda^\varepsilon)^{-1}\| \le \frac{1}{\rho(\lambda^\varepsilon, \sigma(A_0))} \le C,$$

where ρ is the distance function. Hence, since w^ε solves (4.4.10),

$$\|w^\varepsilon\|_{L^2(\Omega)} = \|(B^\varepsilon - \lambda^\varepsilon)^{-1} f^\varepsilon\|_{L^2(\Omega)} \le C\|f^\varepsilon\|_{L^2(\Omega)} \le C.$$

Furthermore

$$\int_{\Omega_0^\varepsilon} |\varepsilon\nabla w^\varepsilon|^2\ \mathrm{d}x = \int_\Omega \lambda^\varepsilon (w^\varepsilon)^2 + f^\varepsilon w^\varepsilon\ \mathrm{d}x \le C\|w^\varepsilon\|_{L^2(\Omega)} \le C.$$

By standard two-scale convergence arguments, we see $w^\varepsilon \xrightharpoonup{2} w(x,y) \in [L^2(\Omega; H)]^d$, $\varepsilon\nabla w^\varepsilon \xrightharpoonup{2} \nabla_y w(x,y)$. Passing to the two-scale limit in (4.4.10) gives (4.4.11). □

Lemma 4.4.2 tells us in particular that for small enough ε the bottom of the spectrum of A^ε is a positive distance away from zero. Taking into consideration the Spectral Theory for self-adjoint operators we can see that a consequence of this is the following Poincaré type inequality:

Lemma 4.4.4. *There exists a constant $C > 0$, independent of ε, such that for all $u \in [H_0^1(\Omega)]^d$ and small enough ε*

$$\|u\|^2_{L^2(\Omega)} \leq C \left(\|C^{(1)}(x/\varepsilon)\nabla u\|^2_{L^2(\Omega_1^\varepsilon)} + \varepsilon^2 \|C^{(0)}(x/\varepsilon)\nabla u\|^2_{L^2(\Omega_0^\varepsilon)} \right). \tag{4.4.12}$$

Proof. Since the bilinear form

$$\beta(u,v) := \int_\Omega \left(C^{(1)}(x/\varepsilon)\nabla u(x) \cdot \nabla v(x) + \varepsilon^2 C^{(0)}(x/\varepsilon)\nabla u(x) \cdot \nabla v(x) \right) \,\mathrm{d}x$$

defines a non-negative quadratic form on the Hilbert space $[H_0^1(\Omega)]^d$, there exists a corresponding self-adjoint operator A^ε such that on a dense subset $\mathcal{D}(A^\varepsilon)$ of $[H_0^1]^d$

$$\beta(u,v) = (A^\varepsilon u, v)_{L^2(\Omega)}.$$

By the classical Rayleigh variational principle,

$$\beta(u,u) = (A^\varepsilon u, u) \geq \lambda_0^\varepsilon (u,u),$$

where λ_0^ε is the smallest eigenvalue of A^ε. Therefore, by the Spectral compactness Lemma, Lemma 4.4.2, we can choose ε small enough such that λ_0^ε is arbitrarily close to $\lambda_0 > 0$, the smallest eigenvalue of the limit spectrum $\sigma(A^0)$. Hence

$$\beta(u,u) \geq c(u,u),$$

for some $c > 0$ and for all $u \in \mathcal{D}(A^\varepsilon)$. This above inequality holds on $[H_0^1]^d$ by the fact $\mathcal{D}(A^\varepsilon)$ is dense in $[H_0^1]^d$. Hence, we have proven (4.4.12). □

Chapter 5

Electromagnetism

In this chapter we shall study the propagation of electromagnetic waves through periodic dielectric materials called photonic crystals. The study of the propagation of waves down a photonic crystal is an important in applications: for example in the creation of electromagnetic waveguides and Photonic crystal fibres. A Photonic crystal is a periodic dielectric material with the dielectric constant varying between material's constitutive parts. If, for a particular photonic crystal, certain frequencies of electromagnetic waves are prohibited to propagate in a given direction then the photonic crystal is said to have a photonic band gap. Mathematically, the photonic band gaps correspond to gaps in the essential spectrum associated with the corresponding Maxwell's equations. Photonic crystals with photonic band gaps are good candidates for electromagnetic waveguides: by sending waves forbidden to propagate through the crystal down a drilled core, one expects the wave to be confined to the core, thus guiding the wave, see Figure 5.1. Mathematically, isolated eigenvalues that appear in the spectral gaps, when perturbing the problem by drilling the core, have eigenfunctions that exponentially decay outside the core. These eigenvalues correspond to the frequencies of guided waves. Motivated by these observations, one would like to know when a photonic crystal has band gaps.

To study the existence of photonic band gaps one asks: for what frequencies ω can one find electromagnetic wave solutions $\hat{E}(x,t) = e^{-i\omega t}E(x)$ and $\hat{H}(x,t) =$

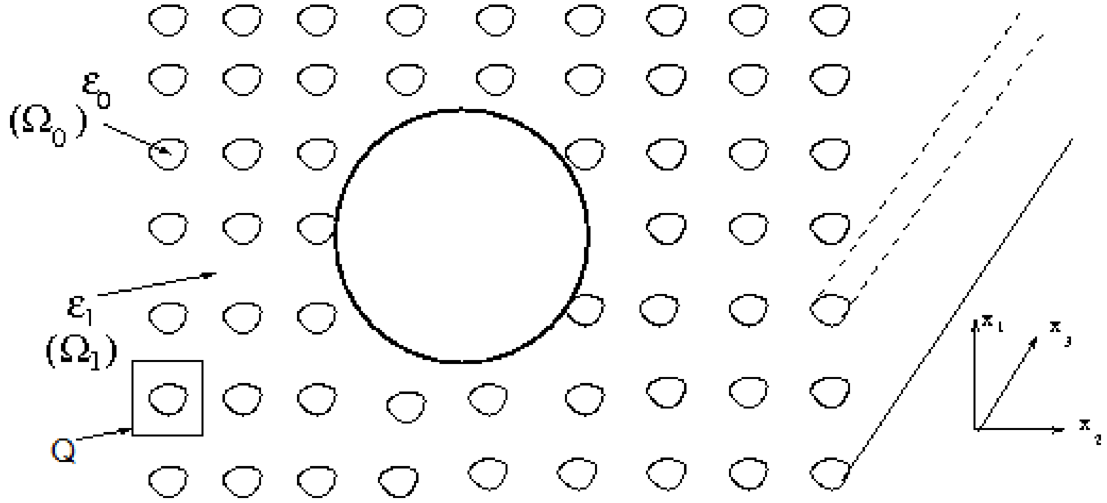

Figure 5.1:

$e^{-i\omega t}H(x)$ to the time-harmonic Maxwell's equation

$$\nabla \times H(x) = -i\omega\varepsilon(x)E(x),$$
$$\nabla \times E(x) = i\omega\mu(x)H(x).$$

Here $\hat{E}$ is the electric field, $\hat{H}$ is the magnetic field multiplied by the free-space impedance, the dielectric permittivity ε and magnetic permeability μ are piecewise constant functions. If ω is in the spectrum, associated to the Maxwell's equations, then the electromagnetic wave propagates through the photonic crystal.

In this chapter we shall consider a photonic fibre that occupies the whole space $\mathbb{R}^3$ that is considered periodic, and heterogeneous in the (x_1, x_2)-plane and homogeneous in the x_3-direction, see Figure 5.2. We shall consider waves propagating in the x_3 direction, i.e. $E = e^{ikx_3}\tilde{E}(x_1, x_2), H = e^{ikx_3}\tilde{H}(x_1, x_2)$ with wave number k chosen to be arbitrarily close to a given 'critical' parameter imposed by the system. In this setting, Maxwell's equations can be reformulated as a PDE system of a partially degenerating type, see Chapter 3. We shall prove that the key assumption, (3.1.20), does hold and has a well defined *non-standard* two-scale homogenised limit. We also prove, in this setting, that the spectral compactness result holds. Therefore, if there exist gaps in the spectrum

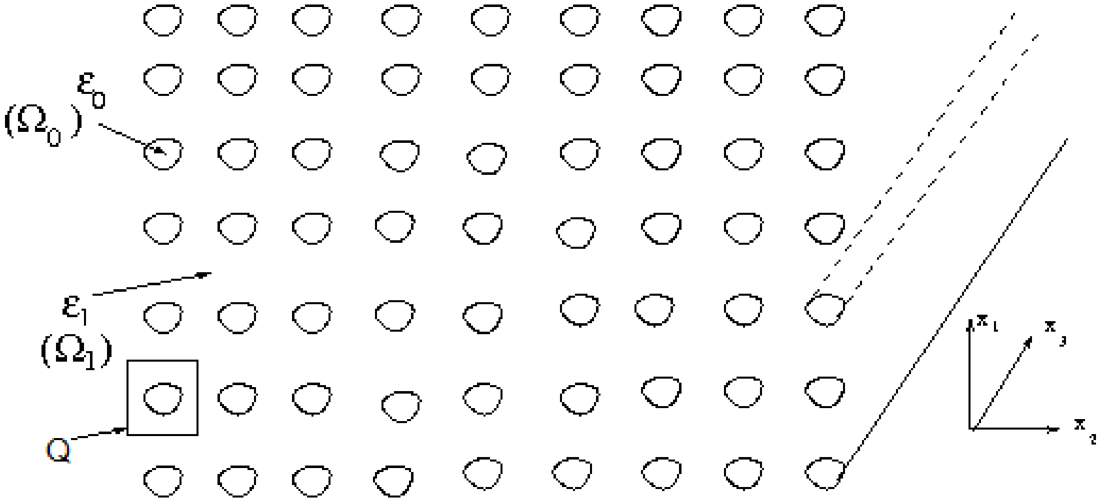

Figure 5.2:

associated with the two-scale limit problem then the photonic fibre has photonic band gaps for a certain range of k.

We shall in turn prove there exist gaps in the limit spectrum for two particular examples: a one-dimensionally periodic 'multilayer' photonic crystal and a two-dimensionally periodic two-phase photonic crystal with the inclusion phase consisting of arbitrarily small balls.

5.1 Problem formulation

Let $Q := [0,1)^2$ be the periodic reference cell, Q_0 be an open bounded subset of Q with sufficiently smooth boundary Γ and $Q_1 := Q \backslash \overline{Q_0}$. Denote by $\tilde{Q} := [0,1)^2 \times \mathbb{R}$, $\tilde{Q}_0 := [0,1)^2 \times \mathbb{R}$ and $\tilde{Q}_1 := \tilde{Q} \backslash \overline{\tilde{Q}_0}$.

We seek solutions $(\tilde{E}, \tilde{H})$ to

$$\begin{aligned} \nabla \times \tilde{H} &= -i\omega\epsilon\tilde{E} \\ \nabla \times \tilde{E} &= i\omega\mu\tilde{H}, \end{aligned} \tag{5.1.1}$$

where the electric permittivity ϵ and the magnetic permeability μ are considered to be piecewise constant functions of the form

$$\tilde{\epsilon} = \epsilon_0 \tilde{\chi}_0(x) + \epsilon_1 \tilde{\chi}_1(x) \qquad\qquad \tilde{\mu} = \mu_0 \tilde{\chi}_0(x) + \mu_1 \tilde{\chi}_1(x),$$

for $i = 1, 2$, where $\tilde{\chi}_i$ is the characteristic functions of $\tilde{Q}_i$.

We look for solutions to (5.1.1) propagating in the x_3 direction with wave number k: $\tilde{E} = e^{ikx_3} E(x_1, x_2), \tilde{H} = e^{ikx_3} H(x_1, x_2)$. Maxwell's equations (5.1.1) are now of the form

$$H_{3,2} - ikH_2 = -i\omega\tilde{\epsilon}E_1 \tag{5.1.2}$$

$$ikH_1 - H_{3,1} = -i\omega\tilde{\epsilon}E_2 \tag{5.1.3}$$

$$H_{2,1} - H_{1,2} = -i\omega\tilde{\epsilon}E_3 \tag{5.1.4}$$

$$E_{3,2} - ikE_2 = i\omega\tilde{\mu}H_1 \tag{5.1.5}$$

$$ikE_1 - E_{3,1} = i\omega\tilde{\mu}H_2 \tag{5.1.6}$$

$$E_{2,1} - E_{1,2} = i\omega\tilde{\mu}H_3. \tag{5.1.7}$$

Rearranging (5.1.5) in for E_2, (5.1.6) for E_1 and substituting into (5.1.3) and (5.1.2) respectively gives

$$(\omega^2\tilde{\epsilon}\tilde{\mu} - k^2)H_1 = ikH_{3,1} - i\omega\tilde{\epsilon}E_{3,2}, \tag{5.1.8}$$

$$(\omega^2\tilde{\epsilon}\tilde{\mu} - k^2)H_2 = ikH_{3,2} + i\omega\tilde{\epsilon}E_{3,1}. \tag{5.1.9}$$

Likewise, re-arranging (5.1.5) for H_1, (5.1.6) for H_2 and substituting into (5.1.3) and (5.1.2) respectively gives

$$(\omega^2\tilde{\epsilon}\tilde{\mu} - k^2)E_1 = ikE_{3,1} + i\omega\tilde{\mu}H_{3,2}, \tag{5.1.10}$$

$$(\omega^2\tilde{\epsilon}\tilde{\mu} - k^2)E_2 = ikE_{3,2} - i\omega\tilde{\mu}H_{3,1}. \tag{5.1.11}$$

Now substituting (5.1.8) and (5.1.9) into (5.1.4), and similarly (5.1.10) and

(5.1.11) into (5.1.7), we arrive at

$$\partial_1\left(\frac{ik}{\tilde{a}}H_{3,2}\right)-\partial_2\left(\frac{ik}{\tilde{a}}H_{3,1}\right)+\partial_1\left(\frac{i\omega\tilde{\epsilon}}{\tilde{a}}E_{3,1}\right)+\partial_2\left(\frac{i\omega\tilde{\epsilon}}{\tilde{a}}E_{3,2}\right)=-i\omega\tilde{\epsilon}E_3 \tag{5.1.12}$$

$$\partial_1\left(\frac{ik}{\tilde{a}}E_{3,2}\right)-\partial_2\left(\frac{ik}{\tilde{a}}E_{3,1}\right)-\partial_1\left(\frac{i\omega\tilde{\mu}}{\tilde{a}}H_{3,1}\right)-\partial_2\left(\frac{i\omega\tilde{\mu}}{\tilde{a}}H_{3,2}\right)=i\omega\tilde{\mu}H_3, \tag{5.1.13}$$

for $\tilde{a} := \omega^2\tilde{\epsilon}\tilde{\mu} - k^2$. Notice that if E_3, H_3 solve (5.1.12)-(5.1.13), substituting these solutions into (5.1.8)-(5.1.11) gives rise to functions $\tilde{E} = e^{ikx_3}E(x_1,x_2)$, $\tilde{H} = e^{ikx_3}H(x_1,x_2)$ which solve the original Maxwell's equations (5.1.1). That is solving (5.1.12)-(5.1.13) is equivalent to solving (5.1.1).

For the remainder of this chapter we shall consider the equations (5.1.12)-(5.1.13). Setting $u = (E_3, H_3)$, problem (5.1.12)-(5.1.13) can be written in the following weak formulation: Find $u \in [H^1(\mathbb{R}^2)]^2$

$$\mathcal{A}(u,\phi) = \int_{\mathbb{R}^2} \omega\rho(x)u\cdot\overline{\phi}\,\mathrm{d}x \quad \forall\phi\in[C_0^\infty(\mathbb{R}^2)]^2. \tag{5.1.14}$$

Here the symmetric bilinear form $\mathcal{A} : [H^1(\mathbb{R}^2)]^2 \times [H^1(\mathbb{R}^2))]^2 \to \mathbb{R}$ is defined by

$$\mathcal{A}(u,\phi) := \int_{\mathbb{R}^2}\frac{\omega}{a}\Big(\epsilon\nabla u_1\cdot\overline{\nabla\phi_1}+\mu\nabla u_2\cdot\overline{\nabla\phi_2}\Big)+\frac{k}{a}\Big(\{u_1,\overline{\phi_2}\}+\{\phi_1,\overline{u_2}\}\Big)\,\mathrm{d}x. \tag{5.1.15}$$

Here $\{v,w\} := v_{,1}w_{,2} - w_{,1}v_{,2}$ (the 'Poisson's bracket'), $a := \omega^2\epsilon\mu - k^2$, and

$$\epsilon := \epsilon_0\chi_0(x)+\epsilon_1\chi_1(x) \qquad\qquad \mu = \mu_0\chi_0(x)+\mu_1\chi_1(x), \tag{5.1.16}$$

and

$$\rho(x) = \chi_0(x)\begin{pmatrix}\epsilon_0 & 0\\ 0 & \mu_0\end{pmatrix} + \chi_1(x)\begin{pmatrix}\epsilon_1 & 0\\ 0 & \mu_1\end{pmatrix}.$$

For $i = 1,2$, χ_i is the characteristic function of Q_i. It can be readily shown that if ω, k satisfy

$$k^2 < \omega^2\min\{\epsilon_0,\epsilon_1\}\min\{\mu_0,\mu_1\}, \tag{5.1.17}$$

then $\mathcal{A}$ is a positive form: there exists a constant $\nu > 0$ such that

$$\mathcal{A}(u,u) \geq \nu \|u\|^2_{H^1(\mathbb{R}^2)}, \quad \forall u \in H^1(\mathbb{R}^2). \tag{5.1.18}$$

A shorthand reformulation of the bilinear form $\mathcal{A}$ is

$$\mathcal{A}(u,v) = \int_{\mathbb{R}^2} \frac{\omega}{a} A \nabla u \cdot \overline{\nabla v}\, \mathrm{d}x = \int_{\mathbb{R}^2} \frac{\omega}{a} A_{ijpq} u_{p,q} \overline{v}_{i,j}\, \mathrm{d}x, \tag{5.1.19}$$

where, for $i,j,p,q = 1,2$, $A_{ijpq} = A^{ip}_{jq}$, and

$$A^{11} = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}, \quad A^{12} = \begin{pmatrix} 0 & \frac{k}{\omega} \\ -\frac{k}{\omega} & 0 \end{pmatrix}, \quad A^{21} = \begin{pmatrix} 0 & -\frac{k}{\omega} \\ \frac{k}{\omega} & 0 \end{pmatrix}, \quad A^{22} = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}. \tag{5.1.20}$$

Note that A is symmetric, i.e. $A_{ijpq} = A_{pqij}$. In future we will represent all fourth order tensors, visually, as follows

$$A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}.$$

All second order tensors, $\{B\}_{ij} = b_{ij}$, shall be visualised as $B = (b_{11}, b_{12}, b_{21}, b_{22})^T$. Under this representation, the tensor contraction $C = AB$ becomes the standard matrix-vector multiplication

$$\begin{pmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{pmatrix} = \begin{pmatrix} A^{11}_{11} & A^{11}_{12} & A^{12}_{11} & A^{12}_{12} \\ A^{11}_{21} & A^{11}_{22} & A^{12}_{21} & A^{12}_{22} \\ A^{21}_{11} & A^{21}_{12} & A^{22}_{11} & A^{22}_{12} \\ A^{21}_{21} & A^{21}_{22} & A^{22}_{21} & A^{22}_{22} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{pmatrix}.$$

From this point we shall consider a 'non-magnetic' dielectric, i.e. to have constant magnetic permeability $\mu_0 = \mu_1 = \mu$ in (5.1.16). Furthermore, assume in (5.1.16) that $\epsilon_0 > \epsilon_1$. We shall consider problem (5.1.19) for the wave number k being close to the critical value given by (5.1.17). That is, taking $k^2 = \omega^2\mu(\epsilon_1 - \varepsilon^2)$, and denoting $\lambda := \omega^2\mu$, (5.1.14) takes the form: Find $u \in [H^1(\mathbb{R}^2)]^2$ such that

$$\int_{\mathbb{R}^2} A(x)\nabla u\cdot\overline{\nabla\phi}\ \mathrm{d}x = \varepsilon^2\lambda\int_{\mathbb{R}^2}\rho(x)u\cdot\overline{\phi}\mathrm{d}x \quad \forall\phi\in[C_0^\infty(\mathbb{R}^2)]^2, \tag{5.1.21}$$

where $A(x)=\chi_1(x)a^{(1)}+\frac{\varepsilon^2}{\epsilon_0-\epsilon_1+\varepsilon^2}\chi_0(x)a^{(0)}$, for

$$(a^{(1)})^{11}=\begin{pmatrix}\epsilon_1 & 0\\ 0 & \epsilon_1\end{pmatrix},\quad (a^{(0)})^{11}=\begin{pmatrix}\epsilon_0 & 0\\ 0 & \epsilon_0\end{pmatrix},\quad (a^{(1)})^{22}=(a^{(0)})^{22}=\begin{pmatrix}\mu & 0\\ 0 & \mu\end{pmatrix},$$

$$(a^{(0)})^{12}=-(a^{(0)})^{21}=(a^{(1)})^{12}=-(a^{(1)})^{21}=\begin{pmatrix}0 & \sqrt{\mu(\epsilon_1-\varepsilon^2)}\\ -\sqrt{\mu(\epsilon_1-\varepsilon^2)} & 0\end{pmatrix}.$$

Note that for this particular choice of k (5.1.17) holds for any small ε, and therefore implies (5.1.18) for any small ε.

Under the ε contraction of the periodic reference cell Q, finding a non-trivial solution, for fixed λ, to (5.1.21) is the same as to find non-trivial solutions to the following problem: Find $u\in[H^1(\mathbb{R}^2)]^2$ such that

$$\int_{\mathbb{R}^2} A(x/\varepsilon)\nabla u\cdot\overline{\nabla\phi}\ \mathrm{d}x = \lambda\int_{\mathbb{R}^2}\rho(x/\varepsilon)u\cdot\overline{\phi}\mathrm{d}x \quad \forall\phi\in[C_0^\infty(\mathbb{R}^2)]^2. \tag{5.1.22}$$

Later, we shall see

$$\int_{\mathbb{R}^2} A(x/\varepsilon)\nabla u\cdot\overline{\nabla\phi}\ \mathrm{d}x = \int_{\mathbb{R}^2}\left[\tilde{a}^{(1)}(x/\varepsilon)+\varepsilon^2\tilde{a}^{(0)}(x/\varepsilon)\right]\nabla u\cdot\overline{\nabla\phi}\ \mathrm{d}x + o(\varepsilon^2),$$

for

$$\tilde{a}^{(1)}(y)=\chi_1(y)\begin{pmatrix}\epsilon_1 & 0 & 0 & \sqrt{\epsilon_1\mu}\\ 0 & \epsilon_1 & -\sqrt{\epsilon_1\mu} & 0\\ 0 & -\sqrt{\epsilon_1\mu} & \mu & 0\\ \sqrt{\epsilon_1\mu} & 0 & 0 & \mu\end{pmatrix}.$$

Scaling $v=(\sqrt{\epsilon_1}u_1,\sqrt{\mu}u_2)$, $\varphi=(\sqrt{\epsilon_1}\phi_1,\sqrt{\mu}\phi_2)$, we have $\tilde{a}^{(1)}\nabla u\cdot\nabla\phi = a^{(1)}\nabla v\cdot\nabla\varphi$

for

$$a^{(1)}(y) = \chi_1(y) \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, (5.1.22) is 'equivalent' to a partially degenerating PDE system with degenerate tensor $a^{(1)}(y)$.

We end this section with the following remark: For particular wave number k close to the critical value $\omega^2\epsilon_1\mu$, the question of whether or not there exist gaps in the spectrum for the operator defined by problem (5.1.14) is equivalent to studying the existence of gaps in the spectrum of the partially degenerating PDE (5.1.22). We shall study such a spectrum of A^ε using the theory of homogenisation.

5.2 Homogenisation

In this section we shall be concerned with seeking the two-scale homogenised limit of the following resolvent problem:
Find $u^\varepsilon \in [H^1(\mathbb{R}^2)]^2$ such that

$$\int_{\mathbb{R}^2} A_\varepsilon(x)\nabla u^\varepsilon \cdot \overline{\nabla\phi}\,\mathrm{d}x + \alpha \int_{\mathbb{R}^2} \rho_\varepsilon(x) u^\varepsilon \overline{\phi}\,\mathrm{d}x = \int_{\mathbb{R}^2} \rho_\varepsilon(x) f^\varepsilon(x)\overline{\phi}\,\mathrm{d}x \quad \forall \overline{\phi} \in \left[C_0^\infty(\mathbb{R}^2)\right]^2. \tag{5.2.1}$$

Here $\alpha > 0$, f^ε known , $\rho_\varepsilon(x) = \rho(x/\varepsilon)$ where

$$\rho(y) = \chi_1(y) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \chi_0(y) \begin{pmatrix} \frac{\epsilon_0}{\epsilon_1} & 0 \\ 0 & 1 \end{pmatrix}; \tag{5.2.2}$$

$A_\varepsilon(x) = A(x/\varepsilon)$ where $A(y) = \chi_1(y)A^{(1)} + \frac{\varepsilon^2}{\epsilon_0-\epsilon_1+\varepsilon^2}\chi_0(y)A^{(0)}$, for

$$A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & \sqrt{1-\frac{\varepsilon^2}{\epsilon_1}} \\ 0 & 1 & -\sqrt{1-\frac{\varepsilon^2}{\epsilon_1}} & 0 \\ 0 & -\sqrt{1-\frac{\varepsilon^2}{\epsilon_1}} & 1 & 0 \\ \sqrt{1-\frac{\varepsilon^2}{\epsilon_1}} & 0 & 0 & 1 \end{pmatrix}, \tag{5.2.3}$$

$$A^{(0)} = \begin{pmatrix} \frac{\epsilon_0}{\epsilon_1} & 0 & 0 & \sqrt{1-\frac{\varepsilon^2}{\epsilon_1}} \\ 0 & \frac{\epsilon_0}{\epsilon_1} & -\sqrt{1-\frac{\varepsilon^2}{\epsilon_1}} & 0 \\ 0 & -\sqrt{1-\frac{\varepsilon^2}{\epsilon_1}} & 1 & 0 \\ \sqrt{1-\frac{\varepsilon^2}{\epsilon_1}} & 0 & 0 & 1 \end{pmatrix}. \tag{5.2.4}$$

It can easily be seen that $(A^1+A^0)(y)$ is symmetric and $(A^1+A^0)(y)$ is uniformly positive definite for sufficiently small ε. That is

$$(A^1 + A^0)(y)_{ijpq} = (A^1 + A^0)(y)_{pqij} \qquad \eta \cdot (A^1 + A^0)(y)\eta > \nu|\eta|^2, \tag{5.2.5}$$

for some $\nu > 0$ independent of ε, for all $\eta \in \mathbb{C}^2$, for all $y \in Q$. This implies the existence and uniqueness of solutions to (5.2.1). We shall denote by A^ε the self-adjoint operator corresponding to problem (5.2.1).

Introducing the following tensors

$$a^{(1)}(y) = \chi_1(y) \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \tag{5.2.6}$$

$$a^{(0)}(y) = \frac{1}{2\epsilon_1}\chi_1(y) \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{(\epsilon_0 - \epsilon_1)}\chi_0(y) \begin{pmatrix} \frac{\epsilon_0}{\epsilon_1} & 0 & 0 & 1 \\ 0 & \frac{\epsilon_0}{\epsilon_1} & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \tag{5.2.7}$$

the first important result is

Lemma 5.2.1. *Let u^ε be the solution to* (5.2.1). *Then there exist constants*

$C > 0$, $E > 0$ *such that for all* $\varepsilon < E$

$$\|u^\varepsilon\|_{L^2(\mathbb{R}^2)} \leq C\|f^\varepsilon\|_{L^2(\mathbb{R}^2)}, \tag{5.2.8}$$

$$\|\varepsilon\nabla u^\varepsilon\|_{L^2(\mathbb{R}^2)} \leq C\|f^\varepsilon\|_{L^2(\mathbb{R}^2)}, \tag{5.2.9}$$

$$\|(a^{(1)})^{1/2}(\tfrac{x}{\varepsilon})\nabla u^\varepsilon\|_{L^2(\mathbb{R}^2)} \leq C\|f^\varepsilon\|_{L^2(\mathbb{R}^2)}. \tag{5.2.10}$$

Furthermore

$$\int_{\mathbb{R}^2} A_\varepsilon(x)\nabla u \cdot \overline{\nabla\phi}\ \mathrm{d}x \ = \ \int_{\mathbb{R}^2} \left[a^{(1)}(x/\varepsilon) + \varepsilon^2 a^{(0)}(x/\varepsilon)\right] \nabla u \cdot \overline{\nabla\phi}\ \mathrm{d}x \, + \, o(\varepsilon^2). \tag{5.2.11}$$

Proof. Taking $\phi = u^\varepsilon$ in (5.2.1) gives

$$\int_{\mathbb{R}^2} A_\varepsilon(x)\nabla u^\varepsilon \cdot \overline{\nabla u^\varepsilon}\ \mathrm{d}x + \alpha\int_{\mathbb{R}^2} \rho_\varepsilon(x)u^\varepsilon \cdot \overline{u^\varepsilon}\ \mathrm{d}x = \int_{\mathbb{R}^2} \rho_\varepsilon(x)f^\varepsilon(x) \cdot \overline{u^\varepsilon}\ \mathrm{d}x. \tag{5.2.12}$$

Using (5.2.12), (5.2.2) and the Cauchy-Schwarz inequality we see

$$\begin{aligned}\int_{\mathbb{R}^2} |u^\varepsilon|^2 &\leq C\int_{\mathbb{R}^2} \rho_\varepsilon u^\varepsilon \cdot \overline{u^\varepsilon} \leq C\int_{\mathbb{R}^2} \rho_\varepsilon f^\varepsilon \cdot \overline{u^\varepsilon} \\ &\leq C\int_{\mathbb{R}^2} f^\varepsilon \cdot \overline{u^\varepsilon} \leq C\left(\int_{\mathbb{R}^2} |f^\varepsilon|^2\right)^{1/2}\left(\int_{\mathbb{R}^2} |u^\varepsilon|^2\right)^{1/2},\end{aligned}$$

resulting in (5.2.8). Similarly (5.2.9) holds due to (5.2.12), (5.2.5), the already known (5.2.8) and the observation that for sufficiently small ε

$$\begin{aligned}\frac{\varepsilon^2}{\epsilon_0 - \epsilon_1 + 1}\int_{\mathbb{R}^2} |\nabla u^\varepsilon|^2 &\leq \frac{\nu\varepsilon^2}{\epsilon_0 - \epsilon_1 + 1}\int_{\mathbb{R}^2} [A^1(x/\varepsilon) + A^0(x/\varepsilon)]\nabla u^\varepsilon \cdot \overline{\nabla u^\varepsilon} \\ &\leq \nu\int_{\mathbb{R}^2} \left[A^1(x/\varepsilon) + \frac{\varepsilon^2}{\epsilon_0 - \epsilon_1 + \varepsilon^2}A^0(x/\varepsilon)\right]\nabla u^\varepsilon \cdot \overline{\nabla u^\varepsilon} \\ &= \nu\int_{\mathbb{R}^2} A_\varepsilon\nabla u^\varepsilon \cdot \overline{\nabla u^\varepsilon}.\end{aligned}$$

We further observe that, by (5.2.3), for sufficiently small ε

$$\int_{\mathbb{R}^2} |(a^{(1)})^{1/2}(x/\varepsilon)\nabla u^\varepsilon|^2 = \int_{\mathbb{R}^2} a^{(1)}(x/\varepsilon)\nabla u^\varepsilon \cdot \overline{\nabla u^\varepsilon} \leq c\int_{\mathbb{R}^2} A_\varepsilon(x/\varepsilon)\nabla u^\varepsilon \cdot \overline{\nabla u^\varepsilon}.$$

Therefore (5.2.10) holds by (5.2.12), (5.2.8) and the Cauchy-Schwarz inequality.

It remains to prove (5.2.11). For fixed $\varepsilon > 0$ such that $\varepsilon < \epsilon_1$ and $\varepsilon < (\epsilon_0 - \epsilon_1)$ we know that

$$\left(1 - \frac{\varepsilon^2}{\epsilon_1}\right)^{1/2} = 1 - \frac{\varepsilon^2}{2\epsilon_1} + O(\varepsilon^4),$$
$$\frac{\varepsilon^2}{\epsilon_0 - \epsilon_1 + \varepsilon^2} = \frac{\varepsilon^2}{\epsilon_0 - \epsilon_1} + O(\varepsilon^4).$$

Now for $A(y)$, given by (5.2.3)-(5.2.4), by direct calculation, we can show :

$$A_\varepsilon(y) = a^{(1)}(y) + \varepsilon^2 a^{(0)}(y) + R^\varepsilon(y),$$

where

$$R^\varepsilon(y) = O(\varepsilon^4).$$

□

Let us now introduce the space $V := \{u \in \left[H^1_\#(Q)\right]^2 : a^{(1)}\nabla_y u = 0\}$. Using (5.2.6) we see the space V can be explicitly represented as

$$V = \left\{v \in [H^1_\#(Q)]^2 : \ \mathrm{div}_y(v) = 0 \text{ and } \mathrm{div}_y(v^\perp) = 0 \text{ in } Q_0\right\}. \qquad (5.2.13)$$

Here $^\perp : H^1_\#(Q) \to H^1_\#(Q)$ is the unitary mapping given by $(u_1, u_2) \mapsto (-u_2, u_1)$. Here, $\mathrm{div}_y u = \nabla_y \cdot u = u_{1,1} + u_{2,2}$, $\mathrm{div}_y u^\perp = \nabla_y^\perp \cdot u = u_{1,2} - u_{2,1}$.

Remark. $v \in V$ is 'equivalent' to v solving the conjugate Cauchy-Riemann equations in Q_1: for $z = y_1 + iy_2$ if we define a complex valued function $F(z) = v_1(y) - iv_2(y)$ then F solves the Cauchy-Riemann equations in Q_1 if, and only if, $v \in V$.

We see by Lemma 5.2.1 that $u^\varepsilon(x) \stackrel{2}{\rightharpoonup} u^0(x, y)$ where the properties of $u^0(x, y)$ are given by Lemma 3.1.1. Following the strategy outlined in Chapter 3 for passing to the two-scale limit in (5.2.1) it is sufficient to prove that, for the degenerate tensor (5.2.6), the key assumption (3.1.21) holds:

Lemma 5.2.2. *There exists a constant $c > 0$ such that for any $u \in [H^1_\#(Q)]^2$*

$$\|P_{V^\perp} u\|^2_{H^1(Q)} \le c\left(\|\mathrm{div}_y u\|^2_{L^2(Q_1)} + \|\mathrm{div}_y u^\perp\|^2_{L^2(Q_1)}\right). \qquad (5.2.14)$$

Here $P_{V^\perp}$ is the orthogonal projection on to $V^\perp$, the orthogonal complement to V in H^1.

To prove this lemma we will use the following regularity result:

Lemma 5.2.3. *Let $u \in H^1_\#(Q)$ be a solution of*

$$-\Delta u = f, \tag{5.2.15}$$

where $f \in L^2_\#(Q)$, $\langle f \rangle = 0$. Then $u \in H^2_\#(Q)$. Furthermore, there exists a constant $c > 0$, depending only on Q, such that

$$\|u\|^2_{H^2(Q)} \leq c \left(\langle u \rangle^2 + \|f\|^2_{L^2(Q)} \right). \tag{5.2.16}$$

Proof. For multi-index $n = (n_1, n_2)$, let w_n , λ_n be the normalised eigenfunctions, and eigenvalues of the periodic Laplacian respectively. Then, $\lambda_n = 4\pi^2 |n|^2$, $w_n(y) = e^{2\pi i n \cdot y}$ and $\{w_n\}$ is an orthonormal basis for $L^2_\#(Q)$. Therefore, we can represent u and f as follows:

$$u = \sum_{|n| \geq 0} a_n w_n, \qquad\qquad f = \sum_{|n| \geq 0} b_n w_n.$$

From the hypothesis it is clear that $b_0 = 0$, $c_n = \frac{b_n}{\lambda_n}$ for $|n| \geq 1$, and from the representation of u we see $c_0 = \langle u \rangle$.

Since $w_n(y)$ are of the form $e^{i2\pi n \cdot y}$ we see that

$$\begin{aligned} \|u\|^2_{H^2(Q)} = \sum_{|n| \geq 0} \left(1 + |n|^2\right)^2 |a_n|^2 = &\ \langle u \rangle^2 + \sum_{|n| \geq 1} \left(1 + 2|n|^2 + |n|^4\right) \left| \frac{b_n}{\lambda_n} \right|^2 \\ \leq &\ \langle u \rangle^2 + c \sum_{|n| \geq 1} |b_n|^2. \end{aligned}$$

This gives (5.2.16). □

Proof of Lemma 5.2.2. Fix $u \in [H^1_\#(Q)]^2$. To prove (5.2.14) it is sufficient to construct a function $v \in V$ such that the difference $\|u - v\|^2_{H^1(Q)}$ is bounded by the right hand side of (5.2.14), cf. (3.1.20).

Let $z := (w_{1,1} - w_{2,2}, w_{1,2} + w_{2,1})$ for w_1, w_2 the solutions to

$$\Delta w_1 = \chi_1 \nabla \cdot u - \chi_0 c_1, \qquad -\Delta w_2 = \chi_1 \nabla^{\perp} \cdot u - \chi_0 c_2. \tag{5.2.17}$$

Here c_1, c_2 are chosen such that the existence of w_1 and w_2 is guaranteed, i.e.

$$c_1 = \frac{1}{|Q_0|} \int_{Q_1} \nabla \cdot u \; \mathrm{d}y, \qquad c_2 = \frac{1}{|Q_0|} \int_{Q_1} \nabla^{\perp} \cdot u \; \mathrm{d}y. \tag{5.2.18}$$

Then, by Lemma 5.2.3, w_1, w_2 belong to $H^2_{\#}(Q)$ which implies $v \in H^1_{\#}(Q)$. Furthermore

$$\nabla \cdot z = w_{1,11} - w_{2,21} + w_{1,22} + w_{2,12} = \Delta w_1 = \chi_1 \nabla \cdot u - \chi_0 c_1,$$

and

$$\nabla^{\perp} \cdot z = -w_{1,21} - w_{2,11} + w_{1,12} - w_{2,22} = -\Delta w_1 = \chi_1 \nabla^{\perp} \cdot u - \chi_0 c_2.$$

Therefore $v := u - z$ belongs to V. Using Lemma 5.2.3, (5.2.17), (5.2.18) and Cauchy-Schwarz inequality we find

$$\begin{aligned}
\|u - v\|^2_{H^1(Q)} = \|z\|^2_{H^1(Q)} &\leq \|\tilde{w}\|^2_{H^2(Q)} \\
&\leq c \left(\langle \tilde{w} \rangle^2 + \|\chi \nabla \cdot u - c_1\|^2_{L^2(Q)} + \|\chi \nabla^{\perp} \cdot u - c_2\|^2_{L^2(Q)} \right) \\
&\leq c \left(\langle \tilde{w} \rangle^2 + \int_{Q_1} |\nabla \cdot u|^2 + c_1^2 |Q_0| + \int_{Q_1} |\nabla^{\perp} \cdot u|^2 + c_2^2 |Q_0| \right) \\
&\leq c \left(\langle \tilde{w} \rangle^2 + \|\nabla \cdot u\|^2_{L^2(Q_1)} + \|\nabla^{\perp} \cdot u\|^2_{L^2(Q_1)} \right).
\end{aligned}$$

This gives the desired inequality by observing that without loss of generality we can choose $\langle \tilde{w} \rangle = 0$ (since adding a constant to $\tilde{w}$ does not change z).

□

Following the general scheme of Chapter 3 further, let us now introduce the following spaces

$$W := \left\{ \Psi \in [L^2_{\#}(Q)]^{2 \times 2} : \mathrm{div}_y((a^{(1)}(y))^{1/2} \Psi(y)) = 0 \right\} \tag{5.2.19}$$

and

$$U := \Big\{ u(x,y) \in L^2(\mathbb{R}^2;V) : \exists \xi(x,y) \in L^2(\mathbb{R}^2;W) \text{ such that },$$
$$\forall \Psi(x,y) \in C^\infty(x,y) \in (\mathbb{R}^2;W), \quad \int_{\mathbb{R}^2}\int_Q \xi(x,y)\cdot\Psi(x,y)\, \mathrm{d}x\mathrm{d}y =$$
$$-\int_{\mathbb{R}^2}\int_Q u(x,y)\cdot \mathrm{div}_x\left(\left(a^{(1)}(y)\right)^{1/2}\Psi(x,y)\right)\, \mathrm{d}x\mathrm{d}y \Big\}. \quad (5.2.20)$$

Also we shall introduce the operator $T : U \to L^2$ defined by $Tu = \xi$ for ξ given in (5.2.20), cf. Chapter 3. Since the key assumption does hold, using Lemma 5.2.1 and using Theorem 3.1.7, we find that the two-scale homogenised limit problem of (5.2.1) is: Find $u \in U$ such that

$$\int_{\mathbb{R}^2}\int_Q Tu_0(x,y)\cdot T\phi_0(x,y) + a^{(0)}\nabla_y u_0(x,y)\cdot\nabla_y\phi_0(x,y)\, \mathrm{d}y\mathrm{d}x+$$
$$+\int_{\mathbb{R}^2}\int_Q \alpha\rho(y)u_0(x,y)\phi_0(x,y)\, \mathrm{d}y\mathrm{d}x = \int_{\mathbb{R}^2}\int_Q \rho(y)f(x,y)\phi_0(x,y)\, \mathrm{d}y\mathrm{d}x,$$
$$\forall \phi_0 \in U. \quad (5.2.21)$$

To proceed with our analysis of the two-scale limit problem (5.2.21) we shall now study the properties of T. In fact, we find that, due to the degeneracy $a^{(1)}$ (5.2.6) for any $u \in U$ the generalised flux Tu is zero.

Proposition 5.2.4. *Let $T : U \to L^2$ defined by $Tu = \xi$ for ξ given in* (5.2.20)*. Then*

$$Tu = 0, \quad \forall u \in U.$$

Proof. It is sufficient to prove the following: for all $u, v \in U$

$$\int_{\mathbb{R}^2}\int_Q Tu(x,y)\cdot Tv(x,y)\, \mathrm{d}y\mathrm{d}x = 0.$$

Let us state now, and prove below, the following property of W:

$$\Psi(y)_{11} + \Psi(y)_{22} = 0 \text{ and } \Psi(y)_{12} - \Psi(y)_{21} = 0 \text{ in } Q_1, \quad \forall \Psi(y) \in W. \quad (5.2.22)$$

For fixed $u \in U$, $Tu \in L^2(\mathbb{R}^2;W)$ and (5.2.22) implies $a^{(1)}Tu = 0$. Therefore $(a^{(1)})^{1/2}Tu = 0$, since $a^{(1)} = \frac{1}{\sqrt{2}}(a^{(1)})^{1/2}$. Now for $\phi \in C_0^\infty(\mathbb{R}^2;V)$, $T\phi =$

$(a^{(1)})^{1/2}(\nabla_x\phi(x,y)+\nabla_y\phi_1(x,y))$ for some $\phi_1 \in C^\infty\left(\mathbb{R}^2; W^\perp\right)$. Moreover,

$$\int_{\mathbb{R}^2}\int_Q Tu(x,y)\cdot T\phi(x,y)\ \mathrm{d}y\mathrm{d}x = \int_{\mathbb{R}^2}\int_Q (a^{(1)})^{1/2}Tu(x,y)\cdot\nabla_x\phi(x,y)\ \mathrm{d}y\mathrm{d}x = 0.$$

Hence the proposition follows by the virtue of the fact that $C_0^\infty(\mathbb{R}^2;V)$ is dense in U.

It remains to show (5.2.22). By definition, for fixed $\Psi\in W$

$$\int_{Q_1}\left(\Psi_{11}+\Psi_{22}\right)\left(\phi_{1,1}+\phi_{2,2}\right)+\left(\Psi_{12}-\Psi_{21}\right)\left(\phi_{1,2}-\phi_{2,1}\right)\ \mathrm{d}y = 0,$$
$$\forall\phi\in\left[H^1_\#(Q)\right]^2. \quad (5.2.23)$$

We now prove (5.2.22) as follows: for fixed $\varphi\in L^2_\#(Q)$ let $u\in H^1_\#(Q)$ be a solution to

$$\Delta u = \varphi - \frac{1}{|Q_0|}\chi_0(y)\langle\varphi\rangle.$$

Choosing our test functions in (5.2.23) as $\phi = (u_{,1}, u_{,2})$ gives

$$0 = \int_{Q_1}\left(\Psi_{11}+\Psi_{22}\right)\Delta u\ \mathrm{d}y = \int_{Q_1}\left(\Psi_{11}+\Psi_{22}\right)\varphi\ \mathrm{d}y.$$

Similarly choosing the test functions to be $\phi = (-u_{,2}, u_{,1})$ gives

$$0 = \int_{Q_1}\left(\Psi_{12}-\Psi_{21}\right)\Delta u\ \mathrm{d}y = \int_{Q_1}\left(\Psi_{12}-\Psi_{21}\right)\varphi\ \mathrm{d}y.$$

This proves $\Psi_{11}(y)+\Psi_{22}(y)=0$ and $\Psi_{12}(y)-\Psi_{21}(y)=0$ for $y\in Q_1$, as required. □

Lemma 5.2.2, Proposition 5.2.4 Problem (5.2.21) imply the following homogenisation theorem.

Theorem 5.2.5. *Let $f^\varepsilon \overset{2}{\rightharpoonup} f$ as $\varepsilon\to 0$ then there exists a $u_0\in U$ such that $u_\varepsilon \overset{2}{\rightharpoonup} u_0$ and furthermore u_0 satisfies the following equation*

$$\int_Q a^{(0)}(y)\nabla_y u_0(x,y)\cdot\overline{\nabla_y\phi(y)}+\alpha\rho(y)u_0(x,y)\cdot\overline{\phi(y)}\ \mathrm{d}y = \int_Q \rho(y)f(x,y)\cdot\overline{\phi(y)}\ \mathrm{d}y$$
$$\forall\phi\in V,\ \text{for a.e. } x \text{ in } \Omega. \quad (5.2.24)$$

Problem (5.2.24) is 'equivalent' to solving: Find $u \in V$ such that

$$\int_Q a^{(0)}(y)\nabla_y u_0(y) \cdot \overline{\nabla_y \phi(y)} + \alpha\rho(y)u_0(y) \cdot \overline{\phi(y)} \, \mathrm{d}y = \int_Q \rho(y)f(y) \cdot \overline{\phi(y)} \, \mathrm{d}y \qquad \forall \phi \in V, \quad (5.2.25)$$

for an appropriate $f(y)$. In particular the spectrum of the operator corresponding to (5.2.24) coincides with the spectrum of the operator for (5.2.25).

By denoting A^0 to be the self adjoint operator defined by the bilinear form $\beta : V \times V \to \mathbb{R}$,

$$\beta(u, v) := \int_Q a^{(0)}(y)\nabla_y u_0(y) \cdot \overline{\nabla_y \phi(y)} + \alpha\rho(y)u_0(y) \cdot \overline{\phi(y)} \, \mathrm{d}y,$$

Theorem 5.2.5 and strong resolvent two-scale convergence imply that

$$\lim_{\varepsilon \to 0} \sigma(A^\varepsilon) \supset \sigma(A^0).$$

Furthermore, since $a^{(0)}$ can easily be shown to have the following property: $\exists \nu > 0$ such that

$$a^{(0)}(y)\eta \cdot \eta > \nu|\eta|^2, \quad \forall \eta \in \mathbb{C}^{2\times 2} \text{ such that } \eta_{11} + \eta_{22} = 0, \; \eta_{12} - \eta_{21} = 0,$$

the bilinear form β is coercive on V. In particular, Lax-Milgram Lemma C.0.3 ensures that A^0 has a compact resolvent. This implies $\sigma(A^0)$ is a countable point spectrum. One expects the spectrum of the original operator to have a band structure, that is a non-empty continuous spectrum. As such, the spectral compactness $\sigma(A^\varepsilon) \to \sigma(A^0)$ should not be expected. The limit operator appears incomplete. In fact, it is incomplete, see Section 5.3. The reason for the incompleteness of the limit operator is due to the fact we can always find a subsequence of u^ε, the sequence of solutions to (5.2.1), whose two-scale limit is a non-trivial NQ periodic function for some multi-index N. Non-triviality in this setting means the NQ periodic function is not Q periodic if $N \neq (1, 1)$.

5.3 Quasi-periodic Homogenisation

In the last section we saw that fixing our periodic reference cell to be Q does not appear to be sufficient for finding the limiting spectrum $\lim_{\varepsilon\to 0}\sigma(A^\varepsilon)$. In this section we will first perform two-scale homogenisation on our original problem (5.2.1) but with a periodic reference cell NQ for a fixed multi-index $N=(N_1,N_2)$. Here $NQ := [0,N_1)\times[0,N_2)$.
Let us begin by reviewing the definition of two-scale convergence, but this time for a periodic set NQ for given multi-index N.

Definition 5.3.1. Let u^ε be a bounded sequence in $L^2(\Omega)$. We say u^ε two-scale converges to $u^0(x,y)\in L^2(\Omega;L^2_\#(NQ))$, denoted by $u^\varepsilon \overset{2}{\rightharpoonup} u^0$, if

$$\int_\Omega u^\varepsilon(x)\phi(x)\phi(x/\varepsilon)\,\mathrm{d}x \longrightarrow \frac{1}{N_1N_2}\int_\Omega\int_{NQ} u^0(x,y)\phi(x)\psi(y)\,\mathrm{d}y\mathrm{d}x,$$
$$\forall\phi\in C_0^\infty(\Omega),\forall\psi\in C_\#^\infty(NQ).$$

It is clear that Lemma 5.2.2, Proposition 5.2.4 and Theorem 5.2.5 all hold with Q replaced by NQ, as the changed size of the periodicity cell does not affect the arguments detailed in Section 5.2. Therefore, the sequence u^ε of solutions to (5.2.1) has a two-scale convergent subsequence to some $u_0\in L^2(\Omega\times NQ)$ such that $u_0\in V^0(N)$, where

$$V^0(N) := \left\{v\in H^1_\#(NQ) : \mathrm{div}_y v = 0 \text{ and } \mathrm{div}_y v^\perp = 0 \text{ in } NQ\backslash F_0\right\}.$$

Here F_0 is the 1-periodic extension of Q_0. Furthermore, $u_0\in V^0(N)$ is the unique solution to

$$\int_{NQ} a^{(0)}(y)\nabla_y u_0(y)\cdot\overline{\nabla_y\phi(y)} + \alpha\rho(y)u_0(y)\cdot\overline{\phi(y)}\,\mathrm{d}y = \int_{NQ}\rho(y)f(y)\cdot\overline{\phi(y)}\,\mathrm{d}y$$
$$\forall\phi\in V^0(N). \quad (5.3.1)$$

Denoting $A^0(N)$ to be the self-adjoint operator defined by (5.3.1) we find, by the same reasons as in Section 5.2, that

$$\lim_{\varepsilon\to 0}\sigma(A^\varepsilon)\supset\sigma(A^0(N)). \quad (5.3.2)$$

Definition 5.3.2. For $\theta \in [0,1)^2$, define the map $A(\theta) : V(\theta) \to L^2(Q)$ where $A(\theta)u = f$ has the precise meaning

$$\int_Q a^{(0)}(y)\nabla_y u(y) \cdot \overline{\nabla_y \phi(y)} \, \mathrm{d}y = \int_Q \rho(y) f(y) \cdot \overline{\phi(y)} \, \mathrm{d}y \quad \forall \phi \in V(\theta), \tag{5.3.3}$$

where

$$V(\theta) := \left\{ u \in H^1_\theta(Q) : \ \ \mathrm{div}_y(u) = 0 \text{ and } \mathrm{div}_y(u^\perp) = 0 \text{ in } Q_0 \right\}.$$

Here $H^1_\theta(Q)$ is the space of $H^1(Q)$ functions that are θ-quasi periodic: $\forall m \in \mathbb{Z}^2$ $u(x+m) = e^{i2\pi m\cdot\theta}u(x)$.

The ellipticity of $a^{(0)}$ on $V(\theta)$ implies, by Lax-Milgram Lemma and the compact embedding $H^1 \subset L^2$, that $A(\theta)$ has a compact resolvent. Therefore, the spectrum of $A(\theta)$ is discrete.

For fixed multi-indices $j = (j_1, j_2)$, $N = (N_1, N_2)$, see Section A.2 in Appendix A , let $\left(\lambda(\frac{j}{N}), w\right)$ be an eigenvalue-eigenfunction pair of $A\left(\frac{j}{N}\right)$, i.e. $w \in V\left(\frac{j}{N}\right)$ and

$$\int_Q a^{(0)}(y)\nabla_y w(y) \cdot \overline{\nabla_y \phi(y)} + \alpha\rho(y)w(y) \cdot \overline{\phi(y)} \, \mathrm{d}y = \lambda(\tfrac{j}{N}) \int_Q \rho(y)w(y) \cdot \overline{\phi(y)} \, \mathrm{d}y \quad \forall \phi \in V\left(\tfrac{j}{N}\right). \tag{5.3.4}$$

A $H^1_{j/N}(Q)$ function can be extended in a quasi-periodic fashion to belong to $H^1_\#(NQ)$. Therefore, we can show that $w \in V^0(N)$. Furthermore, for fixed

$\phi \in C^\infty \cap V^0(N)$,

$$\begin{aligned}
\int_{NQ} a^{(0)}(y)\nabla_y w(y)\cdot\overline{\nabla_y\phi(y)}\ \mathrm{d}y \\
&= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\int_{Q+n} a^{(0)}(y)\nabla_y w(y)\cdot\overline{\nabla_y\phi(y)}\ \mathrm{d}y \\
&= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\int_{Q} a^{(0)}(y+n)\nabla_y w(y+n)\cdot\overline{\nabla_y\phi(y+n)}\ \mathrm{d}y \\
&= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\int_{Q} a^{(0)}(y)\nabla_y e^{i2\pi(j/N)\cdot n} w(y)\cdot\overline{\nabla_y\phi(y+n)}\ \mathrm{d}y \\
&= \int_{Q} a^{(0)}(y)\nabla_y w(y)\cdot\overline{\nabla_y\Phi(y)}\ \mathrm{d}y, \qquad (5.3.5)
\end{aligned}$$

where

$$\Phi(y) = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} e^{(-i2\pi(n_1 j_1/N_1+n_2 j_2/N_2))}\phi(y+n). \qquad (5.3.6)$$

Direct calculation shows $\Phi(y) \in C^\infty \cap V\left(\frac{j}{N}\right)$: In the x_1 direction,

$$\begin{aligned}
\Phi(y+e_1) &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} e^{(-i2\pi(n_1 j_1/N_1+n_2 j_2/N_2))}\phi(y+n+e_1) \\
&= e^{i2\pi(j_1/N_1)}\sum_{m_1=1}^{N_1}\sum_{m_2=0}^{N_2-1} e^{(-i2\pi(m_1 j_1/N_1+m_2 j_2/N_2))}\phi(y+m) \\
&= e^{i2\pi(j_1/N_1)}\Phi(y).
\end{aligned}$$

Similar calculations hold in the x_2 direction and for the derivatives. $\Phi(y)$ is in $V\left(\frac{j}{N}\right)$ since $a^{(1)}\nabla_y\phi(y) = 0$ in Q for $\phi \in V^0(N)$ and $a^{(1)}(y) = a^{(1)}(y+m)$.

By (5.3.4), (5.3.5) and (5.3.6) we find

$$\begin{aligned}\int_{NQ} a^{(0)}(y)\nabla_y w(y)\cdot\overline{\nabla_y\phi(y)} &= \int_Q a^{(0)}(y)\nabla_y w(y)\cdot\overline{\nabla_y\Phi(y)}\\ &= \lambda(\tfrac{j}{N})\int_Q \rho(y)w(y)\cdot\overline{\Phi(y)}\,\mathrm{d}y\\ &= \lambda(\tfrac{j}{N})\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\int_Q \rho(y)e^{i2\pi(j/N)\cdot n}w(y)\cdot\overline{(\phi(y+n))}\,\mathrm{d}y\\ &= \lambda(\tfrac{j}{N})\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\int_Q \rho(y+n)w(y+n)\cdot\overline{(\phi(y+n))}\,\mathrm{d}y\\ &= \lambda(\tfrac{j}{N})\int_{NQ}\rho(y)w(y)\cdot\overline{\phi(y)}\,\mathrm{d}y.\end{aligned}$$

That is, $\lambda(\frac{j}{N})$ is in the spectrum of $A^0(N)$, i.e.

$$\sigma\left(A^0\left(N\right)\right)\supset\sigma\left(A\left(\tfrac{j}{N}\right)\right).$$

This, and (5.3.2), imply that, for multi-index j such that $0\le j\le N-\mathbf{1}$,

$$\lim_{\varepsilon\to 0}\sigma(A^\varepsilon)\supset\sigma\left(A\left(\tfrac{j}{N}\right)\right). \tag{5.3.7}$$

Therefore

$$\lim_{\varepsilon\to 0}\sigma(A^\varepsilon)\supset\bigcup_{0\le j\le N-\mathbf{1}}\sigma\left(A\left(\tfrac{j}{N}\right)\right). \tag{5.3.8}$$

We arrive at the following result:

Lemma 5.3.3.

$$\lim_{\varepsilon\to 0}\sigma(A^\varepsilon)\supset\bigcup_{\theta\in[0,1)^2}\sigma(A(\theta)).$$

Proof. Consider problem (5.3.1) for all multi-indices $N\ge\mathbf{1}$. (Here $\mathbf{1}=(1,1)$). This implies (5.3.2),(5.3.7) and (5.3.8) hold for all N. Therefore

$$\lim_{\varepsilon\to 0}\sigma(A^\varepsilon)\supset\lim_{N\to\infty}\bigcup_{0\le j\le N-\mathbf{1}}\sigma\left(A\left(\tfrac{j}{N}\right)\right).$$

Since $\lim_{\varepsilon\to 0}\sigma(A^\varepsilon)$ is closed we have

$$\lim_{\varepsilon\to 0}\sigma(A^\varepsilon) \supset \overline{\lim_{N\to\infty} \bigcup_{0\le j\le N-\mathbf{1}} \sigma\left(A\left(\tfrac{j}{N}\right)\right)}.$$

To prove the lemma it remains to show that

$$\overline{\lim_{N\to\infty} \bigcup_{0\le j\le N-\mathbf{1}} \sigma\left(A\left(\tfrac{j}{N}\right)\right)} = \bigcup_{\theta\in[0,1)^2} \sigma(A(\theta)). \tag{5.3.9}$$

For any $\theta \in [0,1)^2$, $A(\theta)$ has discrete spectrum. Since the rational numbers are dense in $\mathbb{R}$, (5.3.9) is true if the eigenvalues $\lambda_n(\theta)$ of $A(\theta)$ are continuous with respect to θ. In Section 5.4 we prove that the space $V(\theta)$ is continuous in θ. Therefore $\lambda_n(\theta)$ can be shown to be continuous in θ by the min-max principle:

$$\lambda_n(\theta) = \inf_F \sup_{\substack{u\in F\subset V(\theta)\\ \dim F=n}} \frac{\int_Q a^{(0)}(y)\nabla_y u(y)\cdot\nabla_y\overline{u(y)}\ \mathrm{d}y}{\int_Q \rho(y)u(y)\cdot\overline{u(y)}\ \mathrm{d}y}.$$

□

We shall prove in Section 5.4 that by considering, for all $\theta \in [0,1)^2$, the spectra $\sigma(A(\theta))$ that inclusion

$$\lim_{\varepsilon\to 0}\sigma(A^\varepsilon) \subset \bigcup_{\theta\in[0,1)^2} \sigma(A(\theta)) =: \sigma(A).$$

holds. That is, $\sigma(A^\varepsilon)$ converges in the sense of Hausdorff to $\sigma(A)$.

5.4 Spectral compactness

In this section we prove a more delicate inclusion

$$\lim_{\varepsilon\to 0}\sigma(A^\varepsilon) \subset \bigcup_{\theta\in[0,1} \sigma(A(\theta)),$$

that is we show that if $\lambda^\varepsilon \in \sigma(A^\varepsilon)$, $\lambda^\varepsilon \to \lambda^0$ as $\varepsilon \to 0$, then $\lambda^0 \in \sigma(A(\theta))$ for some θ.

For fixed $\lambda^\varepsilon \in \sigma(A^\varepsilon)$, we know that by Bloch decomposition, there exists a θ^ε-quasi periodic eigenfunction u^ε, normalised i.e. $\|u^\varepsilon\|_{L^2(Q)} = 1$, that solves

$$-\text{div}\,(a^\varepsilon(x)\nabla u^\varepsilon(x)) = \lambda^\varepsilon u^\varepsilon(x), \qquad x \in \varepsilon Q. \tag{5.4.1}$$

Upon rescaling $y = \frac{x}{\varepsilon}$, problem (5.4.1) becomes

$$-\text{div}\,\big(\varepsilon^{-2}a^{(1)}(y) + a^{(0)}(y)\nabla u^\varepsilon(y)\big) = \lambda^\varepsilon u^\varepsilon(y), \qquad y \in Q. \tag{5.4.2}$$

Assuming $\lambda^\varepsilon \to \lambda^0$ we have , up to a subsequence, $\theta^\varepsilon \to \theta^0$, it is sufficient to show the corresponding sequence u^ε of solutions to (5.4.2) converges strongly in L^2 to some non-trivial θ^0-quasi periodic function u^0 that solves

$$\int_Q a^{(0)}(y)\nabla_y u^0(y)\cdot\overline{\nabla_y(\phi(y))}\,\mathrm{d}y = \lambda^0\int_Q \rho(y)u^0(y)\cdot\overline{\phi(y)}\,\mathrm{d}y, \quad \forall\phi\in V(\theta^0).$$

Indeed, if this where true, then $\lambda^0 \in \sigma(A(\theta^0))$. This is the main spectral compactness result.

Theorem 5.4.1. *For $\theta^\varepsilon \in [0,1)^2$, $\lambda^\varepsilon \in \sigma(A^\varepsilon)$, let $u^\varepsilon \in [H^1_{\theta^\varepsilon}(Q)]^2$ satisfy $\|u^\varepsilon\|_{L^2} = 1$ and*

$$\int_Q \varepsilon^{-2}a^{(1)}(y)\nabla u^\varepsilon\cdot\overline{\nabla\phi} + a^{(0)}(y)\nabla u^\varepsilon\cdot\overline{\nabla\phi}\,\mathrm{d}y = \lambda^\varepsilon\int_Q \rho(y)u^\varepsilon\overline{\phi}\,\mathrm{d}y, \qquad \forall\phi\in[H^1_{\theta^\varepsilon}(Q)]^2. \tag{5.4.3}$$

Assume $\lambda^\varepsilon \to \lambda^0$, $\theta^\varepsilon \to \theta^0$ as $\varepsilon \to 0$. Then there exists unique $u^0 \in V(\theta^0)$, $u^0 \neq 0$, such that

$$\int_Q a^{(0)}(y)\nabla_y u^0\cdot\overline{\nabla\phi}\,\mathrm{d}y = \lambda^0\int_Q \rho(y)u^0\cdot\overline{\phi}\,\mathrm{d}y, \quad \forall\phi\in V(\theta^0). \tag{5.4.4}$$

Proof. From the hypotheses and choosing test functions $\phi = u^\varepsilon$ in (5.4.3), we have the following estimates: there exists C independent of ε such that

$$\|u^\varepsilon\|_{L^2} = 1 \qquad \|\nabla u^\varepsilon\|_{L^2} \le C \qquad \|\left(a^{(1)}\right)^{1/2}\nabla u^\varepsilon\|_{L^2} \le \varepsilon^2 C.$$

This implies u^ε converges to u^0 weakly in H^1 and therefore strongly in L^2 as ε

tends to 0, whence $\|u^0\|_{L^2} = 1$. Furthermore $\left(a^{(1)}\right)^{1/2} \nabla u^\varepsilon \to 0$ in L^2. Therefore, $u^0 \in V(\theta^0)$ since $u^\varepsilon \rightharpoonup u^0$ in H^1 implies $\left(a^{(1)}\right)^{1/2} \nabla u^\varepsilon \rightharpoonup \left(a^{(1)}\right)^{1/2} \nabla u^0$ in L^2.

It remains to show that u^0 satisfies (5.4.4). To do this it is sufficient to show for any fixed $\phi^0 \in V(\theta^0)$ that there exists $\phi^\varepsilon \in V(\theta^\varepsilon)$ such that $\phi^\varepsilon \to \phi^0$ strongly in H^1 as $\varepsilon \to 0$. If this where true, since $\phi^\varepsilon \in V(\theta^\varepsilon)$, $a^{(1)}(y)\nabla u^\varepsilon \cdot \overline{\nabla \phi^\varepsilon} = 0$ and taking ϕ^ε as a test function in (5.4.3) gives

$$\int_Q a^{(0)}(y)\nabla u^\varepsilon \cdot \overline{\nabla \phi^\varepsilon}\ \mathrm{d}y = \lambda^\varepsilon \int_Q \rho(y) u^\varepsilon \cdot \overline{\phi^\varepsilon}\ \mathrm{d}y.$$

Now

$$\int_Q a^{(0)}(y)\nabla u^\varepsilon \cdot \overline{\nabla \phi^\varepsilon}\ \mathrm{d}y \to \int_Q a^{(0)}(y)\nabla_y u^0 \cdot \overline{\nabla \phi^0}\ \mathrm{d}y$$

as $\varepsilon \to 0$ since $\phi^\varepsilon \to \phi^0$ in H^1, $\nabla u^\varepsilon \rightharpoonup \nabla u^0$ in L^2 and

$$\begin{aligned}\left| \int_Q a^{(0)}(y) \left[\nabla u^\varepsilon \cdot \overline{\nabla \phi^\varepsilon} - \nabla u^0 \cdot \overline{\nabla \phi^0} \right]\ \mathrm{d}y \right| &\leq \|a^{(0)} \nabla u^\varepsilon\|_{L^2} \|\nabla \phi^\varepsilon - \nabla \phi^0\|_{L^2} \\ &\quad + \left| \int_Q \left(\nabla u^\varepsilon - \nabla u^0\right) \cdot a^{(0)}(y) \overline{\nabla \phi^0}\ \mathrm{d}y \right| \to 0.\end{aligned}$$

Similarly

$$\lambda^\varepsilon \int_Q \rho(y) u^\varepsilon \cdot \overline{\phi^\varepsilon}\ \mathrm{d}y \to \lambda^0 \int_Q \rho(y) u^0 \cdot \overline{\phi^0}\ \mathrm{d}y.$$

This implies u^0 solves equation (5.4.4) and since $\|u^0\|_{L^2} = 1$, $u^0 \neq 0$. Therefore to prove the theorem it is sufficient to show: for all fixed $\phi^0 \in V(\theta^0)$ there exists $\phi^\varepsilon \in V(\theta^\varepsilon)$ such that $\phi^\varepsilon \to \phi^0$ strongly in H^1 as $\varepsilon \to 0$. This is achieved by the following lemma. □

Lemma 5.4.2. *For fixed $\phi^0 \in V(\theta^0)$ there exists $\phi^\varepsilon \in V(\theta^\varepsilon)$ such that $\phi^\varepsilon \to \phi^0$ strongly in H^1 as $\varepsilon \to 0$*

Before we prove this lemma, let us study the structure of functions belonging to $V(\theta)$.

Lemma 5.4.3.
(i). For given $\theta \in [0,1)^2 \backslash \{0\}$, $\phi \in V(\theta)$ if, and only if,

$$\phi = \nabla a + \nabla^\perp b$$

where $a, b \in H^2_\theta(Q)$ *satisfy*

$$\Delta a = f_1 \qquad\qquad \Delta b = f_2,$$

for some $f_1 \in L^2(Q)$*,* $f_2 \in L^2(Q)$*, with* $\mathrm{supp} f_1$ *and* $\mathrm{supp} f_2$ *contained in* $\overline{Q_0}$*.*
(ii). For $\theta = 0$*,* $\phi \in V(0) = V$ *if, and only if,*

$$\phi = c + \nabla a + \nabla^\perp b$$

where c *is a constant vector,* $a, b \in H^2_\#(Q)$ *satisfy*

$$\Delta a = f_1 \qquad\qquad \Delta b = f_2,$$

for some $f_1 \in L^2(Q)$*,* $f_2 \in L^2(Q)$*,* $\langle f_1 \rangle = \langle f_2 \rangle = 0$*,with* $\mathrm{supp} f_1$ *and* $\mathrm{supp} f_2$ *contained in* $\overline{Q_0}$*.*

Proof. For both cases the necessity of the condition is easy to show, the only non-trivial thing is to show the sufficiency. Let us first consider case (i). For $\theta \in [0,1)^2 \backslash \{0\}$, fix $\phi \in V(\theta)$ and set $f_1 := \nabla \cdot \phi$, $f_2 := \nabla^\perp \cdot \phi$. It is clear that $f_1, f_2 \in L^2(Q)$ with their support contained in Q_0. Now let $a, b \in H^1_\theta(Q)$ be the unique solutions of

$$\Delta a = f_1 \qquad\qquad \Delta b = f_2.$$

Setting $w := \phi - \nabla a - \nabla^\perp b$, then $w \in [H^1_\theta(Q)]^2$ and $\Delta w = 0$, which implies $w \equiv 0$.

For case (ii), fix $\phi \in V$; then it is clear that $\phi = c + \phi_0$ for some $\phi_0 \in V(Q)$ such that $\langle \phi_0 \rangle = 0$. Now we repeat the above process, that is let $a, b \in H^1_\#(Q)$ be the unique solutions of

$$\Delta a = \nabla \cdot \phi_0, \quad \langle a \rangle = 0, \qquad\qquad \Delta b = \nabla^\perp \cdot \phi_0, \quad \langle b \rangle = 0.$$

As above $\phi_0 = \nabla a + \nabla^\perp b$. □

Lemma 5.4.4. *For fixed* $\theta \in [0,1)^2 \backslash \{0\}$*, let* $u \in H^1_\theta(Q)$ *be the unique solution of*

$$-\Delta u = f, \tag{5.4.5}$$

where $f \in L^2(Q)$. Then $u \in H^2_\theta(Q)$. Furthermore, there exists a constant $c > 0$, independent of θ, such that

$$\|u\|^2_{H^2(Q)} \le c\left(1 + \frac{1}{|\theta|^2}\right)^2 \|f\|^2_{L^2(Q)}. \tag{5.4.6}$$

Proof. The eigenfunctions $w_z(y;\theta) = e^{i2\pi(\theta+z)\cdot y}$ of the θ-quasi periodic Laplacian form an orthonormal basis in L^2. By decomposing u and f in terms of this basis, we have

$$u(y) = \sum_{z\in\mathbb{Z}^2} a_z e^{i2\pi(\theta+z)\cdot y} \qquad\qquad f(z) = \sum_z b_z e^{i2\pi(\theta+z)\cdot y},$$

for known b_z. Now (5.4.5) tells us $a_z = \frac{b_z}{\lambda(z;\theta)}$, where $\lambda(z;\theta) = 4\pi^2|\theta + z|^2$. Since

$$\|u\|_{H^2} \le \sum_{z\in\mathbb{Z}^2} \left(1 + 4\pi^2|\theta + z|^2\right)^2 |a_z|^2 \le \sum_{z\in\mathbb{Z}^2} \left(\frac{1 + 4\pi^2|\theta + z|^2}{4\pi^2|\theta + z|^2}\right)^2 |b_z|^2.$$

For $|z| \ge 1$ we see $\frac{1+4\pi^2|\theta+z|^2}{4\pi^2|\theta+z|^2} \le 2$. So

$$\|u\|_{H^2} \le 4\left(\frac{1 + |\theta|^2}{|\theta|^2}\right)^2 \|f\|_{L^2}.$$

□

Lemma 5.4.5. *Let $a \in H^1_\#(Q)$, $\langle a\rangle = 0$ be the unique solution to*

$$\Delta a + 4\pi i\theta \cdot \nabla a - 4\pi^2|\theta|^2 a = f, \tag{5.4.7}$$

for given $\theta \in [0, 1)^2$, $f \in L^2(Q)$ such that $\langle f\rangle = 0$. Then $a \in H^2_\#(Q)$ and there exists a constant $C > 0$, independent of θ, such that

$$\|a\|_{H^2} \le C\|f\|_{L^2}.$$

Proof. The eigenfunctions $w_z(y) = e^{i2\pi z\cdot y}$ of the periodic Laplacian form an orthonormal basis in L^2. Decomposing a and f in terms of this basis we have

$$a(y) = \sum_{z\in\mathbb{Z}^2} a_z e^{i2\pi z\cdot y} \qquad\qquad f(z) = \sum_{z} b_z e^{i2\pi z\cdot y},$$

for known b_z. By the hypotheses $a_0 = b_0 = 0$. Now (5.4.7) tells us that, for $z \neq 0$, $a_z = -\frac{b_z}{4\pi^2|z+\theta|^2}$. Since $\theta \in [0,1)^2$,

$$\begin{aligned}\|u\|_{H^2}^2 = \sum_{\substack{z\in\mathbb{Z}^2\\ z\neq 0}} \left(1+|2\pi z|^2\right)^2 |a_z|^2 &= \sum_{\substack{z\in\mathbb{Z}^2\\ z\neq 0}} \left(\frac{1+|2\pi z|^2}{4\pi^2|\theta+z|^2}\right)^2 |b_z|^2 \\ \leq \sum_{\substack{z\in\mathbb{Z}^2\\ z\neq 0}} \left(\frac{1+|2\pi z|^2}{4\pi^2|z|^2}\right)^2 |b_z|^2\end{aligned}$$

i.e.$\|u\|_{H^2}^2 \leq 4\|f\|_{L^2}^2$. □

Proof of Lemma (5.4.2). Consider a sequence $\theta^\varepsilon \subset [0,1)^2$ such that $\theta^\varepsilon \to \theta^0$ as $\varepsilon \to 0$. There are two separate cases to consider, the cases $\theta^0 \in (0,1)^2$ and $\theta^0 = 0$.

Let us consider the case $\theta^0 \in (0,1)^2$: assume without loss of generality that $\theta^\varepsilon \in (0,1)^2$. For fixed $\phi^0 \in V(\theta^0)$ we know that, due to Lemma 5.4.3, $\phi^0 = \nabla a + \nabla^\perp b$ for some $a^0, b^0 \in H^2_{\theta^0}$ where

$$\Delta a^0 = f_1 \qquad\qquad \Delta b^0 = f_2.$$

We shall now construct the desired $\phi^\varepsilon \in V(\theta^\varepsilon)$ as follows: Set $\phi^\varepsilon := \nabla a^\varepsilon + \nabla^\perp b^\varepsilon$ where $a^\varepsilon, b^\varepsilon \in H^1_{\theta^\varepsilon}$ solve

$$\Delta a^\varepsilon = f_1 \qquad\qquad \Delta b^\varepsilon = f_2.$$

By Lemma 5.4.3, $\phi^\varepsilon \in V(\theta^\varepsilon)$. It remains to show $\phi^\varepsilon \to \phi^0$ strongly in $H^1(Q)$. To this end, it is sufficient to show that $a^\varepsilon \to a^0$ and $b^\varepsilon \to b^0$ strongly in H^2 as $\varepsilon \to 0$. By defining $\tilde{a}^\varepsilon(y) := e^{-i2\pi(\theta^\varepsilon-\theta^0)\cdot y}\tilde{a}^\varepsilon(y)$, one notices $\tilde{a}^\varepsilon \in H^2_{\theta^0}$ is the unique solution to

$$\Delta\tilde{a}^\varepsilon = f^\varepsilon, \tag{5.4.8}$$

where $f^\varepsilon := e^{i2\pi(\theta^\varepsilon-\theta^0)\cdot y} f_1 - 4\pi i\,(\theta^\varepsilon-\theta^0)\cdot\nabla a^\varepsilon + 4\pi^2|\theta^\varepsilon-\theta^0|^2 a^\varepsilon$. By Lemma 5.4.4

one finds

$$\|\tilde{a}^\varepsilon - a^0\|_{H^2}^2 \leq C\|f^\varepsilon - f_1\|_{L^2}^2. \tag{5.4.9}$$

Furthermore, $\Delta a^\varepsilon = f_1$ and by an application of Lemma 5.4.4 one notices

$$\|a^\varepsilon\|_{H^2}^2 \leq C\left(1 + \frac{1}{|\theta^\varepsilon|^2}\right)^2 \|f_1\|_{L^2}^2 \leq c\|f_1\|_{L^2}^2,$$

where c is independent of ε (since $\theta^\varepsilon \to \theta^0 \neq 0$). Therefore

$$\begin{aligned}\|f^\varepsilon - f_1\|_{L^2}^2 \leq \|2i\left(\theta^\varepsilon - \theta^0\right) \cdot \nabla\tilde{a}^\varepsilon\|_{L^2}^2 \\ + \||\theta^\varepsilon - \theta^0|^2\tilde{a}^\varepsilon\|_{L^2}^2 + \|\left(e^{i(\theta^\varepsilon - \theta^0)\cdot y}\right) f_1\|_{L^2}^2 \longrightarrow 0,\end{aligned}$$

as $\varepsilon \to 0$. Hence, by (5.4.9), $\tilde{a}^\varepsilon \to a^0$ strongly in H^2. Now one can show $a^\varepsilon \to a^0$ strongly in H^2 by noticing

$$\begin{aligned}\|a^\varepsilon - a^0\|_{H^2}^2 &= \|e^{i2\pi(\theta^\varepsilon - \theta^0)\cdot y}\tilde{a}^\varepsilon - a^0\|_{H^2}^2 \\ &\leq \|e^{i2\pi(\theta^\varepsilon - \theta^0)\cdot y}(\tilde{a}^\varepsilon - a^0)\|_{H^2}^2 + \|(e^{i2\pi(\theta^\varepsilon - \theta^0)\cdot y} - 1)a^0\|_{H^2}^2 \\ &= \|(\tilde{a}^\varepsilon - a^0)\|_{H^2}^2 + \|(e^{i2\pi(\theta^\varepsilon - \theta^0)\cdot y} - 1)a^0\|_{H^2}^2,\end{aligned}$$

and the recalling that $e^{i2\pi x}$ is uniformly continuous with respect to x on $[0,1)^2$. Similarly we can show $b^\varepsilon \to b^0$ strongly in H^2. Therefore $\phi^\varepsilon \to \phi^0$ strongly in H^1 as $\varepsilon \to 0$.

Lets now consider the case $\theta^\varepsilon \to 0$, assuming without loss of generality $\theta^\varepsilon \neq 0$. First let us consider $\phi^0 = c \in \mathbb{R}^2$. We shall show that there exists a sequence of θ^ε-quasi periodic functions ϕ^ε that converges to the constant vector c in H^1. This non-trivial fact will require construction of special functions as follows: Denote by $N^\varepsilon \in H_{\theta^\varepsilon}^2(Q)$ the unique solution to

$$\Delta N^\varepsilon = 2\pi|\theta^\varepsilon|\chi_0. \tag{5.4.10}$$

Let $\phi^\varepsilon = c_1^\varepsilon u_1^\varepsilon + c_2^\varepsilon u_2^\varepsilon$, where $u_1^\varepsilon = \nabla N^\varepsilon$, $u_2^\varepsilon = \nabla^\perp N^\varepsilon$ and $c_1^\varepsilon, c_2^\varepsilon$ are constants which are bounded w.r.t. ε that are yet to be determined. We shall show that $\phi^\varepsilon \to \phi^0 = c$ strongly in H^1 for a specially selected $c_1^\varepsilon, c_2^\varepsilon$.

By the representation $N^\varepsilon(y) = e^{i2\pi\theta^\varepsilon\cdot y}M^\varepsilon(y)$, $M^\varepsilon \in H^2_\#(Q)$, we see, by (5.4.10)

$$\Delta M^\varepsilon + 4\pi i\theta^\varepsilon \cdot \nabla M^\varepsilon - 4\pi^2|\theta^\varepsilon|^2 M^\varepsilon = 2\pi|\theta^\varepsilon|\chi_0 e^{-i2\pi\theta^\varepsilon\cdot y}. \tag{5.4.11}$$

Since there exists a constant C^ε and $\tilde{M}^\varepsilon \in H^2_\#(Q)$, $\langle\tilde{M}^\varepsilon\rangle = 0$ such that $M^\varepsilon = C^\varepsilon + \tilde{M}^\varepsilon$, we see from (5.4.11) that $C^\varepsilon = |2\pi\theta^\varepsilon|^{-1}\langle e^{-i2\pi\theta^\varepsilon\cdot y}\rangle_{Q_0}$ and

$$\Delta\tilde{M}^\varepsilon + 4\pi i\theta^\varepsilon \cdot \nabla\tilde{M}^\varepsilon - 4\pi^2|\theta^\varepsilon|^2\tilde{M}^\varepsilon = f^\varepsilon, \tag{5.4.12}$$

where $f^\varepsilon := 2\pi|\theta^\varepsilon|\chi_0 e^{-i2\pi\theta^\varepsilon\cdot y} + 4\pi^2|\theta^\varepsilon|^2 C^\varepsilon$. It is clear $\langle f^\varepsilon\rangle = 0$, therefore, by Lemma 5.4.5, $\|\tilde{M}^\varepsilon\|_{H^2} \le C\|f^\varepsilon\|_{L^2}$ for some $C > 0$ independent of ε. Hence $\tilde{M}^\varepsilon \to 0$ strongly in H^2, since one observes, by the standard dominated convergence theorem, $f^\varepsilon \to 0$ strongly in L^2. Now

$$\begin{aligned}\phi^\varepsilon = ic_1^\varepsilon\frac{\theta^\varepsilon}{|\theta^\varepsilon|}\langle e^{-i2\pi\theta^\varepsilon\cdot y}\rangle_{Q_0}e^{i2\pi\theta^\varepsilon\cdot y} + ic_2^\varepsilon\frac{\theta^{\varepsilon\perp}}{|\theta^\varepsilon|}\langle e^{-i2\pi\theta^\varepsilon\cdot y}\rangle_{Q_0}e^{i2\pi\theta^\varepsilon\cdot y} + \\ + c_1^\varepsilon\nabla\left(e^{i2\pi\theta^\varepsilon\cdot y}\tilde{M}^\varepsilon\right) + c_2^\varepsilon\nabla^\perp\left(e^{i2\pi\theta^\varepsilon\cdot y}\tilde{M}^\varepsilon\right);\end{aligned}$$

we see $\langle e^{-i2\pi\theta^\varepsilon\cdot y}\rangle_{Q_0}e^{i2\pi\theta^\varepsilon\cdot y} \to 1$ uniformly, and $e^{i2\pi\theta^\varepsilon\cdot y}\tilde{M}^\varepsilon \to 0$ strongly in H^2 as $\varepsilon \to 0$. Therefore to show $\phi^\varepsilon \to c$ strongly in H^1 it is sufficient to let $c_1^\varepsilon, c_2^\varepsilon$ solve

$$ic_1^\varepsilon\frac{\theta^\varepsilon}{|\theta^\varepsilon|} + ic_2^\varepsilon\frac{\theta^{\varepsilon\perp}}{|\theta^\varepsilon|} = c,$$

i.e.

$$c_1^\varepsilon = -\frac{i}{|\theta^\varepsilon|}c\cdot\theta^\varepsilon, \qquad c_2^\varepsilon = -\frac{i}{|\theta^\varepsilon|}c\cdot\theta^{\varepsilon\perp}.$$

It remains to consider the case $\theta^0 = 0$, $\phi^0 \in V(0)$, $\langle\phi^0\rangle = 0$. By Lemma 5.4.3 $\phi^0 = \nabla a + \nabla^\perp b$ for some $a, b \in H^2_\#(Q)$ such that

$$\Delta a = f_1, \qquad \Delta b = f_2,$$

for given f_1, f_2. By setting $\phi^\varepsilon = \nabla a^\varepsilon + \nabla^\perp b^\varepsilon$, where $a^\varepsilon, b^\varepsilon \in H^2_{\theta^\varepsilon}(Q)$ solve

$$\Delta a^\varepsilon = f_1, \qquad \Delta b^\varepsilon = f_2,$$

one can show by arguments very similar to those presented above that $\phi^\varepsilon \to \phi^0$ strongly in H^1. Thus the Lemma is proved. □

5.5 On the limit spectrum

In this section we will consider the limit spectrum for two particular examples. For fixed $\theta \in [-\pi, \pi)^2$, the operator $A(\theta)$, introduced in Section 5.3, has a discrete spectrum of eigenvalues converging to infinity, ordered according to multiplicity

$$0 \leq \lambda_1(\theta) \leq \lambda_2(\theta) \leq \lambda_3(\theta) \leq \dots$$

We proved, in Section 5.4, that the spectrum $\sigma(A^\varepsilon)$ converges, in the sense of Hausdorff, to $\sigma(A)$ in the limit $\varepsilon \to 0$. Hence, if there exist gaps in $\sigma(A)$ then, for sufficiently small ε, there exist gaps in $\sigma(A^\varepsilon)$. Notice, see Section 5.3, that

$$\sigma(A^0) = \bigcup_{i=1}^{\infty} [\min_\theta \lambda_i(\theta), \max_\theta \lambda_i(\theta)].$$

The limit spectrum will have a gap if any two adjacent bands do not overlap: that is, if, for some $i \in \mathbb{N}$,

$$\max_\theta \lambda_i(\theta) < \min_\theta \lambda_{i+1}(\theta).$$

Therefore, for parameter θ, we are interested in studying the following eigenvalue problem:
Find $u \in V(\theta)$ such that

$$\int_Q a^{(0)}(y) \nabla_y u(y) \cdot \overline{\nabla_y(\phi(y))} = \lambda(\theta) \int_Q \rho(y) u(y) \cdot \overline{\phi(y)} \, \mathrm{d}y \quad \forall \phi \in V(\theta). \quad (5.5.1)$$

Here,

$$V(\theta) := \left\{ u \in H^1_\theta(Q) : \quad \mathrm{div}_y(u) = 0 \text{ and } \mathrm{div}_y(u^\perp) = 0 \text{ in } Q_0 \right\}. \quad (5.5.2)$$

Now we turn to the first example.

Example 1. Let us consider a one-dimensional multilayer photonic crystal. Namely, consider a slab like inclusion, i.e. $Q_0 = [a, b] \times [0, 1)$, $0 < a < b < 1$. Seek a solution of the form $u_0(x, y) = u(x, y_1)e^{iky_2}$. Then (5.5.2) implies that $u_0 \in V(\theta)$, $\theta = (\theta_1, \theta_2)$, if u satisfies

$$\frac{\partial u_1}{\partial y_1} + iku_2 = 0, \quad \text{in } Q_0 \tag{5.5.3}$$

$$iku_1 - \frac{\partial u_2}{\partial y_1} = 0, \quad \text{in } Q_0. \tag{5.5.4}$$

That yields $u(x, y_1) = a_1(x) \begin{pmatrix} -ie^{ky_1} \\ e^{ky_1} \end{pmatrix} + a_2(x) \begin{pmatrix} ie^{-ky_1} \\ e^{-ky_1} \end{pmatrix}$. Heuristically, we expect that propagation is more likely to be forbidden in direction of the greatest dielectric discontinuity, therefore let us consider waves that propagate in the direction with the greatest variation in electric permitivity, i.e. set $k = 0$ and $\theta_2 = 0$. Equations (5.5.3)-(5.5.4) imply that $u \in V(\theta)$ if u is θ-quasi periodic and is a constant vector in Q_0, i.e. $u \in V(\theta)$ if $u \in [H^1_\theta[0, 1]]^2$, and u is a constant vector in $[a, b]$.

We now wish to study problem (5.5.1), which takes the following form: For fixed $\theta, \lambda(\theta)$, find $u = (u_1, u_2) \in V(\theta)$ such that

$$\int_{[0,a)\cup(b,1]} \begin{pmatrix} 2u_1' \\ u_2' \end{pmatrix} \cdot \begin{pmatrix} \overline{\phi_1'} \\ \overline{\phi_2'} \end{pmatrix} = \lambda(\theta) \int_0^1 \rho u \cdot \overline{\phi} \, \mathrm{d}y, \quad \forall \phi \in V(\theta). \tag{5.5.5}$$

Here

$$\rho(y) = \chi_0(y) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \chi_1(y) \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

(We have chosen $\varepsilon_0 = 2$, $\varepsilon_1 = 1$ for simplicity). Notice that, since $\phi \in C_0^\infty([0, 1]\backslash[a, b])$ is an admissible test function, (5.5.5) implies that

$$-u''(y) = \lambda(\theta)u(y) \qquad y \in [0, a) \cup (b, 1]. \tag{5.5.6}$$

Furthermore, since $u \in V(\theta)$, $u(y) = C \in \mathbb{C}^2$ in $[a, b]$. Therefore, integrating by parts in (5.5.5), and using (5.5.6), gives

$$\lambda(\theta) \int_a^b C \cdot \overline{\phi} \, \mathrm{d}y = 2u_1'(y)\overline{\phi_1}(y)|_{y=b}^a + u_2'(y)\overline{\phi_2}(y)|_{y=b}^a, \quad \forall \phi \in V(\theta). \tag{5.5.7}$$

$V(\theta)$ is two-dimensional and is spanned by functions that are, in $[a, b]$, of the form $(1, 0)^T$ and $(0, 1)^T$ respectively. Therefore (5.5.7) can be reduced to the following algebraic system:

$$\lambda(\theta)(b-a)C_1 = 2u_1^{'}(a) - 2u_1^{'}(b),$$
$$\lambda(\theta)(b-a)C_2 = u_2^{'}(a) - u_2^{'}(b).$$

Taking all of this into consideration, we see that solving (5.5.5) is equivalent to simultaneously finding u and C such that

$$-u^{''}(y) = \lambda(\theta)u(y) \quad y \in [0, a) \cup (b, 1], \tag{5.5.8}$$
$$u(1) = e^{i2\pi\theta}u(0), \qquad u'(1) = e^{i2\pi\theta}u'(0), \tag{5.5.9}$$

with the following interface conditions

$$u(a) = u(b) = C \tag{5.5.10}$$
$$\lambda(\theta)(b-a)C_1 = 2u_1^{'}|_{y=b}^{a}, \tag{5.5.11}$$
$$\lambda(\theta)(b-a)C_2 = u_2^{'}|_{y=b}^{a}. \tag{5.5.12}$$

Seeking solutions to (5.5.8)-(5.5.12) of the form

$$u(y) = A^1 e^{i\sqrt{\lambda(\theta)}(\theta)x} + A^2 e^{-i\sqrt{\lambda(\theta)}(\theta)x} \qquad y \in [0, a),$$
$$u(y) = B^1 e^{i\sqrt{\lambda(\theta)}(\theta)x} + B^2 e^{-i\sqrt{\lambda(\theta)}(\theta)x} \qquad y \in (b, 1],$$

for some $A^1 = (A_1^1, A_2^1)^T$, $A^2 = (A_1^2, A_2^2)^T$, $B^1 = (B_1^1, B_2^1)^T$, $B^2 = (B_1^2, B_2^2)^T$ to be determined, we arrive at a linear system that can be represented as follows: find $X = (C, A^1, A^2, B^1, B^2)^T$ such that

$$M(\lambda(\theta), \theta)X = 0,$$

where $M(\lambda(\theta)$ is the corresponding 10×10 matrix. To solve this we find the zeros of the function $F : \mathbb{R} \times [-\pi, \pi) \to \mathbb{C}$ defined by $F(\lambda, \theta) := \det M(\lambda, \theta)$. The λ, in set of the zeros of F, make up the spectrum of the limit operator, see Figure 5.3. We find that the limit spectrum of the one-dimensional crystal does indeed have gaps, therefore by the spectral compactness result of the previous section

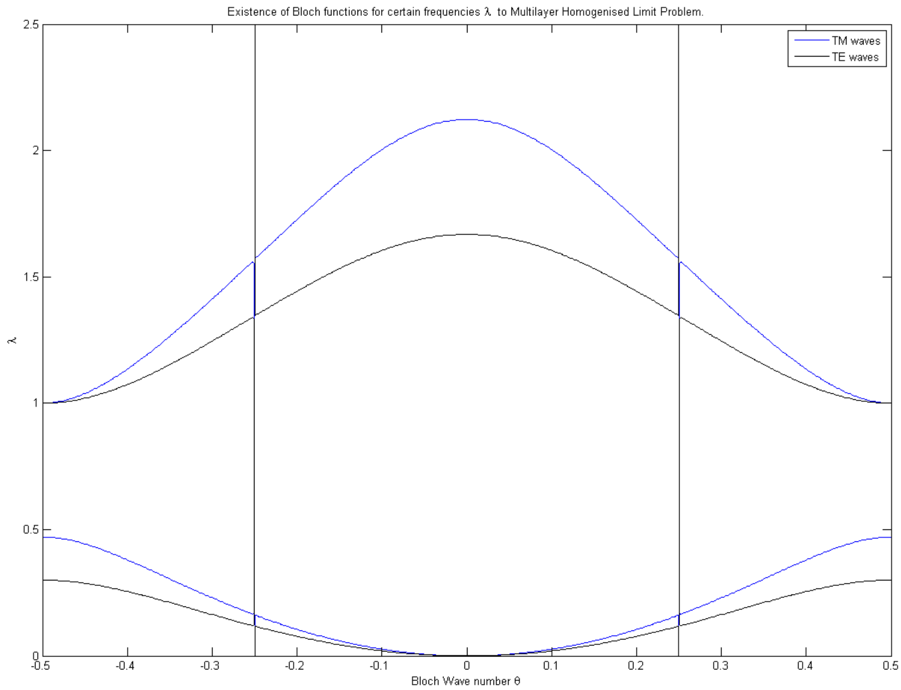

Figure 5.3: Band gap structure of limit spectrum. Plot was made by using Matlab to find the level curve $F(\lambda, \theta) = 0$.

we know that, for small enough ε, the original spectrum also contains gaps.

Remark. In Figure 5.3 we see two curves whose image is a part of the spectrum of our homogenised limit operator. The reason for this is because due to the symmetry of the domain in x_1-direction, waves propagating with wave number $(k_{x_1}, k_{x_2}) = (0, k)$ are polarised. These are the transverse magnetic (TM) and transverse Electric (TE) polarisations. The presence of such polarisations implies that we could have decomposed problem (5.1.21) into finding solutions of the form $u^\varepsilon = (u_1^\varepsilon, 0)$ and $u^\varepsilon = (0, u_2^\varepsilon)$ respectively. These functions two-scale converge to functions of the form $(u_1, 0)$ and $(0, u_2)$ respectively, therefore leading to an effective TM and TE polarisation of the limit problem. It is to be noted here that the general two-dimensional case has no such polarisations and the solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ to problem (5.1.21) cannot be decoupled. This makes the two-dimensional case intrinsically more difficult to study. We consider a two-dimensional example below, with small inclusions, and prove there exists at least one spectral gap using variational arguments, cf. [20].

Example 2. We shall now consider a photonic fibre with a two-dimensional periodic structure. In particular, we consider a photonic fibre with small circular inclusions. That is, let $Q_0 = B_\delta(0)$ the ball of radius δ centered at the origin. Here $\delta << 1$ is a new small parameter. Such a photonic fibre could be created, for example, by drilling a periodic array of cylindrical holes of a very small cross-sectional radius in a dielectric material and then filling the holes with a material that is more optically dense than the background dielectric ($\varepsilon_0 > \varepsilon_1$). (Note in passing that such models are known in physics as ARROW fibres, see e.g. [24].)

We will show that for sufficiently small δ the limit spectrum $\sigma(A^0)$ has at least one gap. To this end, we will show that there exists constants $c_1, c_2 > 0$ such that

$$\lambda_2(\theta) \leq -\frac{c_2}{\delta^2 \ln \delta}, \qquad \lambda_3(\theta) \geq c_1 \delta^{-2}. \tag{5.5.13}$$

This implies that, for small enough δ, there is a spectral gap in the limit spectrum, and therefore also for small enough ε for the original problem by the spectral compactness.

Let us now show that (5.5.13) holds. For $\theta = 0$ we see that $\lambda(0) = 0$ is an eigenvalue, with multiplicity 2, for (5.5.1): the orthogonal eigenfunctions, of $\lambda(0) = 0$, are $u_0(y) = (1, 0)$ and $v_0(y) = (0, 1)$. This implies

$$\lambda_1(0) = \lambda_2(0) = 0.$$

Let us now consider the more interesting, non-trivial case $\theta \neq 0$. By classical variational arguments, it is known that

$$\lambda_1(\theta) = \min_{\substack{u \in V(\theta) \\ u \neq 0}} \frac{a(u, u)}{(\rho u, u)},$$

where

$$a(u, v) := \int_Q a^{(0)}(y) \nabla_y u(y) \cdot \overline{\nabla_y v(y)} \, \mathrm{d}y.$$

By uniform ellipticity and boundedness of $a^{(0)}$ and ρ, it is clear that there exists a constant c such that

$$\lambda_1(\theta) \le c \min_{\substack{u\in V(\theta)\\ u\neq 0}} \frac{\int_Q |\nabla u|^2}{\int_Q |u|^2}.$$

We shall now show that the right hand side of the above inequality is bounded from above by $-\frac{c_2}{\delta^2 \ln \delta}$. Denote $u = \nabla N$ where $N \in H^2_\theta$ is the solution to

$$-\Delta N = \chi_0,$$

which exists and is unique for $\theta \neq 0$. Then

$$\int_Q |\nabla u|^2 \le \int_Q |\nabla^2 N| = \int_Q |\Delta N|^2 = \delta^2 |B|,$$

and

$$\int_Q |u|^2 = \int_Q |\nabla N|^2 = -\int_Q \Delta N \cdot N = \int_{Q_0} N \ge -c\delta^4 \ln \delta \tag{5.5.14}$$

The last inequality is due to subtle technical arguments, see Proposition 5.5.1, Proposition 5.5.2 and Corollary 5.5.3 below. Hence, $\lambda_1(\theta) \le -\frac{c_2}{\delta^2 \ln \delta}$. Furthermore, this result also follows for $v = \nabla^\perp N$ and, since u, v are orthogonal, the min-max variational principle implies $\lambda_2(\theta) \le -\frac{c_2}{\delta^2 \ln \delta}$.

We shall next show that the second inequality in (5.5.13) for $\lambda_3(\theta)$ holds for $\theta \neq 0$ and therefore will hold for $\theta = 0$ by continuity of $\lambda(\theta)$. Fix $\theta \in (-\pi/2, \pi/2]^2 \backslash \{0\}$. By variational principle and ellipticity of $a^{(0)}$ and ρ,

$$\lambda_3(\theta) \ge c \inf_{\substack{u\perp v\\ u\perp w}} \frac{\int_Q |\nabla u|^2}{\int_Q |u|^2},$$

for some constant c and, for any $v, w \in V(\theta)\backslash\{0\}$ such that $v \perp w$. Here $\perp$ should be read as orthogonality in L^2. We shall choose $v = \nabla N$, $w = \nabla^\perp N$ for N constructed above; clearly $v \perp w$. For fixed $u \in V(\theta)$, by Lemma 5.4.3,

$u = \nabla a + \nabla^{\perp} b$ for some $a, b \in H^2_\theta$ that are harmonic in Q_1. Note that, for $u \perp v$

$$0 = \int_Q u \cdot \overline{v} = \int_Q \begin{pmatrix} a_{,1} - b_{,2} \\ a_{,2} + b_{,1} \end{pmatrix} \cdot \begin{pmatrix} \overline{N_{,1}} \\ \overline{N_{,2}} \end{pmatrix} = -\int_Q a \overline{\Delta N} = \int_{Q_0} a,$$

which implies $\langle a \rangle_{Q_0} = 0$. Similarly $u \perp w$ implies $\langle b \rangle_{Q_0} = 0$. Furthermore, we observe that

$$\frac{\int_Q |\Delta a|^2}{\int_Q |\nabla a|^2} = \frac{\int_Q |\Delta a|^2}{-\int_Q \Delta a \cdot \overline{a}} \geq \mu_2 \delta^{-2}, \quad \mu_2 > 0. \tag{5.5.15}$$

Indeed, notice that

$$\begin{aligned} \left| \int_Q \Delta a \cdot \overline{a} \right| &\leq \left(\int_{Q_0} |\Delta a|^2 \right)^{1/2} \left(\int_{Q_0} |a|^2 \right)^{1/2} \\ &\leq \left(\int_{Q_0} |\Delta a|^2 \right)^{1/2} (\delta^{-2}\mu_2)^{-1/2} \left(\int_{Q_0} |\nabla a|^2 \right)^{1/2} \\ &\leq \left(\delta^{-2}\mu_2\right)^{-1/2} \left(\int_{Q_0} |\Delta a|^2 \right)^{1/2} \left(\int_{Q} |\nabla a|^2 \right)^{1/2} \\ &= \left(\delta^{-2}\mu_2\right)^{-1/2} \left(\int_{Q_0} |\Delta a|^2 \right)^{1/2} \left(-\int_{Q} \Delta a \cdot \overline{a} \right)^{1/2}. \end{aligned} \tag{5.5.16}$$

The second inequality is due to the following Poincaré type inequality:

$$\int_{Q_0} |a|^2 \leq \mu_2^\delta \int_{Q_0} |\nabla a|^2, \tag{5.5.17}$$

where μ_2^δ is the first non-zero eigenvalue of the Neumann Laplacian on $Q_0 = B_\delta(0)$. The inequality (5.5.17) can be shown by a simple application of the spectral theory for self-adjoint operators. Furthermore, by a simple rescaling argument, $\mu_2^\delta = \delta^{-2}\mu_2$, where μ_2 is the first non-zero eigenvalue of the Neumann Laplacian on the unit ball, $B_1(0)$. Then (5.5.15) immediately follows from (5.5.16).

Similarly

$$\frac{\int_Q |\Delta b|^2}{\int_Q |\nabla b|^2} \geq \mu_2 \delta^{-2}.$$

Furthermore, since $a, b \in H^2_\theta(Q)$, then by integration by parts

$$\int_Q \nabla a \cdot \overline{\nabla^\perp b} = \int_Q -a_{,1}\overline{b_{,2}} + a_{,2}\overline{b_{,1}} = \int_Q a\overline{b_{,21}} - a\overline{b_{,12}} = 0,$$
$$\int_Q \nabla b \cdot \overline{\nabla^\perp a} = \int_Q -b_{,1}\overline{a_{,2}} + b_{,2}\overline{a_{,1}} = \int_Q b\overline{a_{,21}} - b\overline{a_{,12}} = 0.$$

This implies, for $u = \nabla a + \nabla^\perp b$,

$$\int_Q |u|^2 = \int_Q |\nabla a|^2 + \int_Q |\nabla^\perp b|^2 = \int_Q |\nabla a|^2 + \int_Q |\nabla b|^2.$$

Since, for any a, b, by a similar direct inspection,

$$\int_Q \nabla^2 a \cdot \overline{\nabla\left(\nabla^\perp b\right)} = \int_Q \nabla^2 b \cdot \overline{\nabla\left(\nabla^\perp a\right)} = 0,$$

we also obtain

$$\int_Q |\nabla u|^2 = \int_Q |\Delta a|^2 + \int_Q |\Delta b|^2.$$

All of these considerations, and (5.5.15), imply that

$$\frac{\int_Q |\nabla u|^2}{\int_Q |u|^2} = \frac{\int_Q |\Delta a|^2}{\int_Q |\nabla a|^2 + \int_Q |\nabla b|^2} + \frac{\int_Q |\Delta b|^2}{\int_Q |\nabla a|^2 + \int_Q |\nabla b|^2} \geq \delta^{-2}\mu_2 \frac{\int_Q |\nabla a|^2 + \int_Q |\nabla b|^2}{\int_Q |\nabla a|^2 + \int_Q |\nabla b|^2}.$$

Hence $\lambda_3(\theta) \geq \delta^{-2}\mu_2$.

Finally, we prove (5.5.14).

Proposition 5.5.1. *Let $Q_0 = B_\delta(0) \subset \mathbb{R}^2$ be the open ball of radius $\delta < \frac{1}{2}$ centred at the origin. For fixed $\theta \in [-\pi, \pi)^2 \setminus \{0\}$, let $u \in H^1_\theta(Q)$ be the unique solution to*

$$-\Delta u = \chi_0. \tag{5.5.18}$$

Then

$$\int_{B_\delta(0)} u = \frac{\pi}{8}\delta^4 - \pi^2\delta^4\left(\frac{1}{2\pi}\ln\delta - g_0(\theta)\right), \tag{5.5.19}$$

for some constant $g_0(\theta)$ depending on θ.

Proposition 5.5.2. *Let $g_0(\theta)$ be given by Proposition 5.5.1. Then $g_0(\theta)$ is uniformly bounded from above with respect to θ: there exists a constant $A \in \mathbb{R}$*

independent of θ *such that*

$$g_0(\theta) \geq A, \quad \forall \theta \in [-\pi, \pi)^2 \backslash \{0\}\,.$$

Corollary 5.5.3. *There exists a* δ_0 *and a constant* $C > 0$ *such that for all* $\delta < \delta_0$, *and all* $\theta \in [-\pi, \pi) \backslash \{0\}$

$$\int_{B_\delta(0)} u \geq -C\delta^4 \ln \delta.$$

Proof of Proposition (5.5.1). Denote by $G^\theta(x)$ the θ-quasi periodic Green's function for $-\Delta$ on Q with its singularity at the origin. It is known that, by isolating the singularity and expanding in Fourier series in θ,

$$G^\theta(x) = -\frac{1}{2\pi} \ln r + g_0(\theta) + \sum_{n=1}^{\infty} g_n^{\pm} r^n e^{in\varphi}, \quad \text{for } x \in B_\delta(0). \tag{5.5.20}$$

Here $g_0(\theta)$ and $g_n^{\pm}$ are known constants, r, φ are the polar coordinates. We shall now construct an explicit solution to (5.5.18). Define u as follows:

$$u = \begin{cases} \pi\delta^2 G^\theta(x), & \text{in } Q\backslash B_\delta(0) \\ -\frac{r^2}{4} + B + \pi\delta^2 \left(G^\theta(x) + \frac{1}{2\pi}\ln r - g_0(\theta)\right), & \text{in } B_\delta(0). \end{cases} \tag{5.5.21}$$

The constant B is chosen such that $u \in H^1_\theta(Q)$, i.e.

$$B = \frac{\delta^2}{4} - \pi\delta^2 \left(\frac{1}{2\pi} \ln \delta - g_0(\theta)\right).$$

We will now show that u solves (5.5.18). For fixed $\varphi \in H^1_{\#}(Q)$

$$\int_Q \nabla u \cdot \nabla \varphi = -\int_{Q\backslash B_\delta(0)} \Delta u \varphi - \int_{B_\delta(0)} \Delta u \varphi + \int_{\partial B_\delta(0)} \left[\frac{\partial u}{\partial n}\right] \varphi, \tag{5.5.22}$$

where n is unit outward normal to $B_\delta(0)$, $\frac{\partial u}{\partial n} := \nabla u \cdot n$ and $\left[\frac{\partial u}{\partial n}\right]$ is the jump across the interface $\partial B_\delta(0)$. From (5.5.20) and (5.5.21) we notice

$$-\int_{Q\setminus B_\delta(0)} \Delta u\ \varphi = 0, \tag{5.5.23}$$

$$-\int_{B_\delta(0)} \Delta u\ \varphi = -\int_{B_\delta(0)} \Delta(-\frac{r^2}{4})\varphi = \int_{B_\delta(0)} \varphi = \int_Q \chi_0\ \varphi, \tag{5.5.24}$$

$$\begin{aligned}\left[\frac{\partial u}{\partial n}\right] &= \frac{\partial}{\partial r}\left(\pi\delta^2 G^\theta - \left[-\frac{r^2}{4} + B + \pi\delta^2\left(G^\theta + \frac{1}{2\pi}\ln r - g_0(\theta)\right)\right]\right)\Big|_{r=\delta} \\ &= \frac{\partial}{\partial r}\left[\frac{r^2}{4} - \frac{\delta^2}{2}\ln r\right]\Big|_{r=\delta} = 0. \end{aligned} \tag{5.5.25}$$

Equations (5.5.23)-(5.5.25) and (5.5.22) imply that u solves (5.5.18). It remains to show (5.5.19):

$$\begin{aligned}\int_{B_\delta(0)} u &= \int_{B_\delta(0)} \left[-\frac{r^2}{4} + B + \pi\delta^2\left(G^\theta(x) + \frac{1}{2\pi}\ln r - g_0(\theta)\right)\right] \\ &= \int_{B_\delta(0)}\left[-\frac{r^2}{4} + B\right] + \pi\delta^2 \underbrace{\int_{B_\delta(0)} \sum_{n=1}^{\infty} g_n^{\pm} r^n e^{in\varphi}}_{=0} \\ &= -\frac{\pi}{8}\delta^4 + \pi\delta^2 B = \frac{\pi\delta^4}{8} - \pi^2\delta^4\left(\frac{1}{2\pi}\ln\delta - g_0(\theta)\right).\end{aligned}$$

□

Proof of Proposition (5.5.2). Consider $G(\theta;k;x)) \in H^1_\theta(Q\setminus\{0\})$ such that

$$G(\theta;k;x) = -\frac{1}{2\pi}\ln r + g_0(\theta,k) + o(r\ln r) \quad \text{as } r \to 0,$$

and

$$(-\Delta - k)\,G = 0, \quad \text{in } Q\setminus\{0\}. \tag{5.5.26}$$

This function is well defined, for example, if $k < |\theta|^2$, which could be seen from explicit analysis of the eigenvalues of $-\Delta$. We aim to show that there exists a constant A such that

$$g_0(\theta, 0) \geq A, \quad \forall \theta \in [-\pi, \pi)^2, \tag{5.5.27}$$

since $G(\theta; 0; x)$ is nothing more than $G^\theta(x)$, the Green's function used in the proof of Proposition 5.5.1. Note that, for fixed negative k, e.g. $k = -1$, $g_0(\theta; -1)$ is continuous with respect to $\theta \in [-\pi, \pi)^2$ and therefore is uniformly bounded from below by some constant A, i.e.

$$g_0(\theta, -1) \geq A, \quad \forall \theta \in [-\pi, \pi)^2,$$

Moreover, $g_0(\theta; k)$ is continuous with respect to $k \in [-1, 0]$. Therefore, to prove (5.5.27) it is sufficient to prove

$$g_0'(\theta; k) > 0, \quad k \in [-1, 0), \tag{5.5.28}$$

where the prime, $'$, denotes differentiation with respect to k.

We shall now prove (5.5.28). Differentiating (5.5.26) with respect to k gives

$$(-\Delta - k)\, G' - G = 0, \quad \text{in } Q\backslash 0 \tag{5.5.29}$$

and

$$G'(\theta; k; x) = g_0'(\theta, k) + \ldots \quad \text{as } r \to 0. \tag{5.5.30}$$

Multiplying (5.5.29) by $\overline{G}$ and integrating over $Q\backslash B_\delta(0)$, for sufficiently small δ, gives

$$\begin{aligned}
\int_{Q\backslash B_\delta(0)} G\overline{G} &= \int_{Q\backslash B_\delta(0)} (-\Delta - k)\, G'\overline{G} = \int_{\partial B_\delta(0)} \left(\frac{\partial}{\partial n} G'\overline{G} - G' \frac{\partial}{\partial n}\overline{G} \right) \\
&= \int_{\partial B_\delta(0)} \left[(O(1) + \ldots) \left(-\frac{1}{2\pi} \ln r + \ldots \right) - (g_0'(\theta, k) + \ldots) \left(-\frac{1}{2\pi r} + \ldots \right) \right] \\
&= \int_{\partial B_\delta(0)} g_0'(\theta, k) \frac{1}{2\pi r} + o(\delta \ln \delta),
\end{aligned}$$

passing to the limit $\delta \to 0$ gives

$$\int_Q |G|^2 = g_0'(\theta, k).$$

This proves (5.5.28). $\square$

Chapter 6

Further work

We briefly discuss here some further questions caused by this work. The decoupling in the two-scale limit problem for the partially degenerate elasticity and the resulting absence of band gaps in Chapter 4 are specific for the chosen partial degeneracy. We expect more general elastic partially degenerate problems to remain coupled (cf. [35, 15, 2]), and it would be interesting to both analyse these from the point of view of the general theory (Chapter 3), and to describe wider classes when the decoupling does occur.

From Chapter 5 we conclude that the θ-quasi periodic dependence of the two-scale homogenised limit, in the case $\Omega = \mathbb{R}^d$, is a consequence of the degeneracy $a^{(1)}$ not only being non-negative in the matrix Q_1 but this degeneracy is such that the space $V(\theta)$ is not 'equivalent' to V. Such a condition says nothing about $\partial\Omega$ and as such it is conceivable that the homogenised limit of Dirichlet problems, in domains with a boundary, can be θ dependent. Important questions about the interplay between the quasi-periodic 'Bloch' decomposition of the problem in y and the boundary conditions in x become apparent, e.g.: can the limit spectrum have any band structure, even in the case when the original spectrum is discrete? If such a situation occurs, does the spectral compactness hold? Also, one could ask for what general class of the degeneracies, $a^{(1)}$, the corresponding 'macroscopic' homogenised operator is zero, i.e. $T \equiv 0$. Furthermore, the interplay between T and the θ dependence of the limit needs to be studied: can one get non-trivial θ dependence in the limit problems with $T \neq 0$? Other open questions and possibility for further work from Chapter 5 are: What is the spectrum of the limit operator A^0 for Q_0 of more general shapes and/or sizes? In particular can we determine, either analytically or numerically, whether the band gaps still exist? Questions about the 'spectral compactness' of the isolated eigenvalues of the actual photonic fibre (i.e. with its core) to the isolated eigenvalues of the perturbed limit problem needs to be explored, (cf. [22, 11] for the two-dimensional scalar case).

We also see in Chapter 5 the need to develop the general theory outlined in Chapter 3 further by including the possible θ-dependence of the limit operator. Work to further develop the general theory could be done by studying not only partially degenerating $a^\varepsilon(y)$ but also partially degenerating 'densities' ρ^ε, cf. [3] for a particular example. This could be expected to provide a richer class of two-scale limit behaviours with the hope of applications to wider sets of non-standard effects, in particular for metamaterials, cf. for example [25, 7].

Appendix A

Notation

A.1 Functions and function spaces.

- $\Omega \subset \mathbb{R}^d$ is open and bounded, unless stated otherwise.
- $L^2_{\#}(Q)$ denotes the set of square integrable functions that are Q-periodic a.e.
- $C^\infty_{\#}(Q)$ is the set of Q-periodic infinitely smooth functions.
- For a given set Ω, $C^\infty_0(\Omega)$ denotes the space of infinitely smooth functions whose support is a compact subset of Ω.
- For a given set Ω, $H^1(\Omega)$ is the Sobolev space of square integrable functions with square integrable gradients.
- $H^1_{\#}(Q)$ denotes the set of $H^1_{\text{loc}}(\mathbb{R}^d)$ functions that are Q-periodic a.e. Equivalently $H^1_{\#}(Q)$ is the closure of $C^\infty_{\#}(Q)$ with respect to the standard H^1 norm.
- $H^1_0(\Omega)$ is the set of $H^1(\Omega)$ functions with zero trace. Equivalently $H^1_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ with respect to the standard H^1 norm.
- $[X]^n$ denotes the n-dimensional vector space with each coefficient belonging to the space X, i.e. for $u \in [X]^n$, $u = (u_1, \ldots, u_n)^T$, where $u_j \in X$ for all $j = 1, \ldots, n$.

- $[X]^{n\times d}$ denotes the space of n by d matrices whose components belong to X, i.e. for $A \in [X]^{n\times d}$, $A_{ij} \in X$ for $i = 1, \ldots, n$; $j = 1, \ldots, d$.

- For the Banach space X we denote by $\|\cdot\|_X$ the norm of X. Where it is clear which Banach space we are referring to we drop the suffix on the norm identifying the space.

- $L^2(\Omega; Y)$, $H^1(\Omega; Y)$, $C^\infty(\Omega; Y)$, etc are obvious extensions of the above definitions for functions with values in the Banach space Y.

- For a given function f, $f_{,i}$ denotes the i-th partial derivative of f, i.e

$$f_{,i} := \frac{\partial f}{\partial x_i}.$$

- For repeated indices we use the Einstein summation convention. For example, $A\nabla u \cdot \nabla v = A_{ij} u_{,j} v_{,i}$ means

$$A\nabla u \cdot \nabla v = \sum_{i=1}^{d} \sum_{j=1}^{d} A_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}.$$

- For given function f, $\langle f \rangle_B$ will denote the average of f over B. i.e.

$$\langle f \rangle_B \ := \frac{1}{|B|} \int_B f(y) \ \mathrm{d}y.$$

  Furthermore we shall use the shorthand $\langle f \rangle$ for $\langle f \rangle_Q$.

- For a given set B, χ_B denotes the characteristic function of B. i.e

$$\chi_B(x) = \begin{cases} 1, & x \in B \\ 0, & \text{otherwise.} \end{cases}$$

- We shall use χ_1 and χ_0 as shorthand notation for χ_{Q_1} and χ_{Q_0} respectively.

A.2 Multi-indices

A d-dimensional multi-index α is a d-tuple

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d),$$

where, for $i = 1, \ldots, d$, $\alpha_i \in \mathbb{N}_0$.

- For fixed multi-indices α, β, $\beta_i \neq 0$, we denote $\frac{\alpha}{\beta} := \left(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_d}{\beta_d}\right)$.
- For fixed multi-indices α, β, we denote by $\alpha \cdot \beta$ the sum $\sum_{i=1}^{d} \alpha_i \beta_i$.
- For fixed multi-indices α, β, $\alpha \leq \beta$ if, and only, if $\alpha_i \leq \beta_i \ \forall i = 1, \ldots, d$.

Appendix B

Two-scale convergence

We review here the concept of two-scale convergence and state some of its main properties that are useful in the homogenisation of second order PDEs. For a full account of two-scale convergence and its use in homogenisation see [27],[1],[37].

Definition B.0.1. Let u_ε be a bounded sequence in $L^2(\Omega)$. We say u_ε (weakly) two-scale converges to $u_0 \in L^2(\Omega \times Q)$, denoted by $u_\varepsilon \stackrel{2}{\rightharpoonup} u_0$, if for all $\phi \in C_0^\infty(\Omega)$, $\psi \in C_\#^\infty(Q)$

$$\int_\Omega u_\varepsilon(x)\phi(x)\psi\left(\frac{x}{\varepsilon}\right) \mathrm{d}x \longrightarrow \int_\Omega \int_Q u_0(x,y)\phi(x)\psi(y) \; \mathrm{d}x\mathrm{d}y$$

as $\varepsilon \to 0$.

Let us now mention some properties of two-scale convergence which are of particular use in the application of two-scale convergence to homogenisation theory.

Lemma B.0.2 (**Some properties of two-scale convergence**)**.**

(i) *If u_ε is bounded in $L^2(\Omega)$ then there exists $u_0 \in L^2(\Omega \times Q)$ and a subsequence $u_{\varepsilon'}$ such that $u_{\varepsilon'} \stackrel{2}{\rightharpoonup} u_0$.*

(ii) *If $u_\varepsilon \stackrel{2}{\rightharpoonup} u_0$ then u_ε converges to $\int_Q u_0 \, \mathrm{d}y$ weakly in $L^2(\Omega)$.*

(iii) *If $u(y) \in C_\#^\infty(Q)$ then $u(x/\varepsilon) \stackrel{2}{\rightharpoonup} u(y)$.*

(iv) *If $u_\varepsilon \stackrel{2}{\rightharpoonup} u_0$ and $a(y) \in L_\#^\infty(Q)$ then $a(x/\varepsilon)u_\varepsilon(x) \stackrel{2}{\rightharpoonup} a(y)u_0(x,y)$.*

(v) If $u_\varepsilon \overset{2}{\rightharpoonup} u_0$ *then*

$$\liminf_{\varepsilon\to 0} \int_\Omega u_\varepsilon^2 \ \mathrm{d}x \geq \int_\Omega \int_Q u_0^2 \ \mathrm{d}x\mathrm{d}y.$$

We see, for example, while $\sin(x/\varepsilon) \rightharpoonup 0$, $\cos(x/\varepsilon) \rightharpoonup 0$ weakly in L^2, Lemma B.0.2 (iii) tells us that $\sin(x/\varepsilon) \overset{2}{\rightharpoonup} \sin(y)$ and $\cos(x/\varepsilon) \overset{2}{\rightharpoonup} \cos(y)$. This shows us that, while weak convergence is not particularly useful in telling us the oscillatory structure of rapidly oscillating functions, the two-scale limit keeps information about the structure of the oscillations. Two-scale convergence is only capable of following oscillations on the same order of the test function's oscillations; for example by the mean value property $\sin(x/\varepsilon^2) \overset{2}{\rightharpoonup} \langle\sin(y)\rangle = 0$. Here the two-scale limit tells us nothing of the oscillatory structure of the function $u_\varepsilon := \sin(x/\varepsilon^2)$.

The important relative two-scale compactness property, Lemma B.0.2 (i), was first shown by Nguetseng [27]. Along with the notion of weak two-scale convergence we have the complementary notion of strong two-scale convergence.

Definition B.0.3. A sequence $u_\varepsilon \in L^2(\Omega)$ is said to strongly two-scale converge to $u \in L^2(\Omega \times Q)$, denoted $u_\varepsilon \overset{2}{\to} u$, if $u_\varepsilon \overset{2}{\rightharpoonup} u$ and

$$\int_\Omega u_\varepsilon(x) v_\varepsilon(x) \ \mathrm{d}x \to \int_\Omega \int_Q u(x,y)v(x,y) \ \mathrm{d}x\mathrm{d}y,$$

for all $v_\varepsilon \overset{2}{\rightharpoonup} v$.

By Definition B.0.3, if $u^\varepsilon \overset{2}{\to} u$ then $u^\varepsilon \overset{2}{\rightharpoonup} u$ and

$$\lim_{\varepsilon\to 0} \int_\Omega |u_\varepsilon(x)|^2 \ \mathrm{d}x = \int_\Omega \int_Q |u(x,y)|^2 \ \mathrm{d}y\mathrm{d}x$$

This can in fact be shown to be a sufficient condition for strong two-scale convergence. That is an equivalent useful definition of strong two-scale convergence is:

Lemma B.0.4. u_ε *strongly two-scale converges to* u *if, and only if,* $u_\varepsilon \overset{2}{\rightharpoonup} u$ *and*

$$\lim_{\varepsilon\to 0} \int_\Omega |u_\varepsilon(x)|^2 \ \mathrm{d}x = \int_\Omega \int_Q |u(x,y)|^2 \ \mathrm{d}x\mathrm{d}y.$$

B.1 Two-scale resolvent convergence

If we consider a sequence of non-negative self-adjoint operators A_ε and wish to study their behaviour as $\varepsilon \to 0$, then we study the limit behaviour of their resolvents, i.e. for $\alpha > 0$

$$A_\varepsilon u + \alpha u = f.$$

In homogenisation theory we often find that $u^\varepsilon := (A_\varepsilon + \alpha)^{-1} \in L^2(\Omega)$ two-scale converges to some $u^0(x, y) \in L^2(\Omega \times Q)$ where

$$A_0 u^0 + \alpha u^0 = f,$$

for a self-adjoint operator A^0. The notion of strong resolvent convergence is not applicable here since the limiting operator $(A_0 + \alpha)^{-1}$ is defined on the space $L^2(\Omega \times Q)$ while $(A_\varepsilon + \alpha)^{-1}$ is defined on $L^2(\Omega)$. We instead use the notion of strong two-scale resolvent convergence, see [10, 38].

Definition B.1.1. Let A_ε, A_0 be non-negative self-adjoint operators on $L^2(\Omega)$ and the closed linear subspace $H \subset L^2(\Omega \times Q)$ respectively. Then we say A_ε strong two-scale resolvent converges to A_0, denoted $A_\varepsilon \xrightarrow{2} A_0$ if for some $\alpha > 0$

$$(A_\varepsilon + \alpha)^{-1} f_\varepsilon \xrightarrow{2} (A_0 + \alpha)^{-1} P f \quad \text{whenever} \quad f_\varepsilon \xrightarrow{2} f \in L^2(\Omega \times Q),$$

where $P : L^2(\Omega \times Q) \to H$ is the orthogonal projection.

Remark. Since the resolvent set is an open subset of $\mathbb{C}$, as in the case of strong resolvent convergence, it is sufficient to test strong two-scale resolvent convergence for a single $\alpha > 0$ in the resolvent set, say $\alpha = 1$. Furthermore it can often be shown that strong resolvent convergence is implied by the 'weak' two-scale resolvent convergence :

$$(A_\varepsilon + \alpha)^{-1} f_\varepsilon \overset{2}{\rightharpoonup} (A_0 + \alpha)^{-1} P f \quad \text{whenever} \quad f_\varepsilon \overset{2}{\rightharpoonup} f \in L^2(\Omega \times Q).$$

If A_ε strong two-scale resolvent converges to A_0 then the limit spectrum $\sigma(A_0)$ is always contained in the limiting spectrum $\lim_{\varepsilon \to 0} \sigma(A_\varepsilon)$. That is

Proposition B.1.2. *Let A_ε, A_0 be non-negative self-adjoint operators on $L^2(\Omega)$ and the subspace $H \subset L^2(\Omega \times Q)$ respectively, let $A_\varepsilon \xrightarrow{2} A_0$. Then for all $\lambda_0 \in$*

$\sigma(A_0)$ there exists $\lambda_\varepsilon \in \sigma(A_\varepsilon)$ such that $\lambda_\varepsilon \to \lambda_0$ as $\varepsilon \to 0$.

In general, it is not the case that the reverse inclusion holds. However if this is true, that is, if $\lim_{\varepsilon\to 0} \sigma(A_\varepsilon) \subset \sigma(A_0)$ then the proof of this fact is more difficult and problem specific. It usually requires establishing, by seperate means, a version of 'two-scale spectral compactness'.

Appendix C

Some Functional Analysis facts

Lemma C.0.3 (Lax-Milgram Lemma)**.** *For a Hilbert space H, let $\beta : H \times H \to \mathbb{R}$ be a bilinear form. Assume β is bounded and coercive. i.e. there exist constants $C > 0, \nu > 0$ such that*

$$\beta(u,v) \leq C\|u\|_H \|v\|_H, \qquad \forall u, v \in H,$$
$$\beta(u,u) \geq \nu \|u\|_H^2, \qquad \forall u \in H.$$

Then for fixed f in H^*, the space of bounded linear functionals on H, there exists a unique solution $u \in H$ to

$$\beta(u,v) = < f, v >, \quad \forall v \in H.$$

Furthermore,

$$\|u\|_H \leq \nu^{-1} \|f\|_{H^*}.$$

Lemma C.0.4 (Lion's Lemma, cf. [17])**.** *Let Ω be a bounded open domain with C^1 boundary. Let u be a distribution on Ω such that $u \in H^{-1}(\Omega), u_{,i} \in H^{-1}(\Omega)\ \forall i$. Then $u \in L^2(\Omega)$ and there exists a constant $c > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leq c \left(\|\nabla u\|_{H^{-1}(\Omega)} + \|u\|_{H^{-1}(\Omega)} \right).$$

Lemma C.0.5. *For $u \in H^1_{\#}(Q)$ there exists $\tilde{u}$ and a constant $c > 0$ independent of u such that*

(i) $u = \tilde{u}$ *in* Q_1,

(ii) $\Delta\tilde{u} = 0$ *in* Q_0,

(iii) $\|\nabla\tilde{u}\|_{L^2(Q)} \leq c\|u\|_{H^1_\#(Q_1)}$.

We shall call $\tilde{u}$ *the harmonic extension of* u.

Proof. For fixed $u \in H^1_\#(Q)$, Sobolev Extension theorem, c.f. e.g. [39], says there exists an extension operator $E : H^1_\#(Q_1) \to H^1_\#(Q)$ such that

$$\|Eu\|_{H^1(Q)} \leq c\|u\|_{H^1(Q_1)},$$

for some constant $c > 0$ independent of u. Denote by $\tilde{u} \in H^1(Q)$ the solution to

$$-\Delta\tilde{u} = 0 \quad \text{in } Q_0, \qquad\qquad \tilde{u} = u \quad \text{on } \partial Q_0,$$

extended by u into Q_1; $\tilde{u}$ satisfies (i) & (ii). Since $\tilde{u}$ minimises the functional

$$F(u) = 1/2 \int_{Q_0} |\nabla u|^2 \mathrm{d}x$$

on $\{u + H^1_0(Q_0)\}$ we have, in particular,

$$\int_{Q_0} |\nabla\tilde{u}|^2 \, \mathrm{d}y \leq \int_{Q_0} |\nabla\,(Eu)\,|^2 \, \mathrm{d}y.$$

Inequality (iii) follows from the properties of the extension operator. □

Lemma C.0.6 (Korn's Inequality)**.** *Let* $u \in H^1(\Omega)$. *Then there exists a constant* $c > 0$ *independent of* u *such that*

$$\|u\|^2_{H^1(\Omega)} \leq c\left(\int_\Omega |u|^2 \, \mathrm{d}y + \int_\Omega |e(u)|^2 \, \mathrm{d}y\right)$$

Lemma C.0.7 (Korn's Inequality for periodic functions)**.** *Let* $u \in H^1_\#(Q_1)$, *i.e.* $u \in H^1(Q)$ *and* Q*-periodic for a.e.* $y \in Q_1$. *Then there exists a constant* $c > 0$ *independent of* u *such that*

$$\|w\|^2_{H^1_\#(Q_1)} \leq c\left(\left(\int_{Q_1} w \, \mathrm{d}y\right)^2 + \int_{Q_1} |e(w)|^2 \, \mathrm{d}y\right)$$

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